TAM 212 Worksheet 8: Bicycle gearing

The aim of this worksheet is to understand how different bicycle gearing systems work. Shown below is Prof. Ertekin’s bike, which uses *derailleur gears*.

In contrast, Prof. West’s bike (shown below) uses a *hub gear*, in which the gears are contained within the rear hub itself.

Both Prof. Ertekin’s and Prof. West’s bikes are normally parked outside MEB if you would like a closer look at the gear systems on them, or if you want to investigate how well professors maintain their bikes.
Derailleur gears on Prof. Ertekin’s bike

The basic components of a derailleur bicycle transmission are shown below (not to scale). The crankarm and the chainring are rigidly connected, so they both rotate at the same angular velocity. Similarly, the sprocket is fixed to the rear wheel, so they also both turn at the same angular velocity. The chain cannot stretch, so all points on the chain move with the same speed, which is also the speed of the outer edge of the chainring and sprocket.

1. Taking \( r_c \) to be the radius of the chainring and \( \omega_c \) to be the angular velocity of the crankarm, what is the speed \( v_c \) of each point on the chain relative to the bike? Because we are working relative to the bike, we take the centers \( O_s \) and \( O_c \) to be fixed points.

**Solution:** \( v_c = r_c \omega_c \)

2. Taking \( r_s \) to be the radius of the sprocket, what is the angular velocity \( \omega_s \) of the rear sprocket in terms of the chain speed \( v_c \)?

**Solution:** \( \omega_s = \frac{v_c}{r_s} \)
3. Using your answers above, derive an expression for the ratio $\omega_s/\omega_c$ in terms of the radii $r_s$ and $r_c$.

**Solution:**

$$\frac{\omega_s}{\omega_c} = \frac{r_c}{r_s}$$

4. The rear wheel of the bike has radius $r_w$ and is rigidly connected to the sprocket, so $\omega_w = \omega_s$. Using your previous answer, derive an expression for the speed $v_w$ of the bike in terms of $\omega_c$ and all radii.

**Solution:**

$$v_w = r_w\omega_w$$

$$= r_w\omega_s$$

$$= \frac{r_w r_c}{r_s} \omega_c$$

5. Prof. Ertekin’s bike has Shimano Diore components, with 3 chainrings and 8 sprockets having radii as shown in the table below.

<table>
<thead>
<tr>
<th>Chainring</th>
<th>$r_C$</th>
<th>Teeth (1/2'' pitch)</th>
<th>Sprocket</th>
<th>$r_S$</th>
<th>Teeth (1/2'' pitch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>0.0485 m</td>
<td>24</td>
<td>S1</td>
<td>0.0222 m</td>
<td>11</td>
</tr>
<tr>
<td>C2</td>
<td>0.0647 m</td>
<td>32</td>
<td>S2</td>
<td>0.0263 m</td>
<td>13</td>
</tr>
<tr>
<td>C3</td>
<td>0.0848 m</td>
<td>48</td>
<td>S3</td>
<td>0.0303 m</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>S4</td>
<td>0.0344 m</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>S5</td>
<td>0.0384 m</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>S6</td>
<td>0.0424 m</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>S7</td>
<td>0.0485 m</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>S8</td>
<td>0.0566 m</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>S9</td>
<td>0.0647 m</td>
<td>32</td>
</tr>
</tbody>
</table>

While riding into the office, Prof. Ertekin observes that she is pedaling with a frequency of $f_c = 1$ Hz (so $\omega_c = 2\pi f_c \approx 6.28$ rad/s). Her bike has 700C wheels, so $r_w \approx 0.35$ m. Compute expected min and max speeds using the gear ratios in the table above.

**Solution:**

$$min : C1/S9 : \quad v_w \approx 5.93\text{km/h} = 1.65\text{m/s}$$

$$max : C3/S1 : \quad v_w \approx 30.2\text{km/h} = 8.39\text{m/s}$$
Hub gears on Prof. West’s bike

Prof. West’s bike uses a fixed chainring and sprocket sizes and the gearing is provided in the hub. The internally geared hub looks like the cut-away diagram shown on the left below (this is actually an old Sturmey-Archer design).

The motion transmission mechanism here is a planetary gear, which is arranged as shown to the right above. The outer ring gear rolls on the three planet gears, which in turn roll on the fixed central sun gear. The planet cage joins together pins at the centers of three planet gears and the sun gear.

The following diagram shows the measurements and variables (not to scale). In this diagram, point A is a material point which is attached to the fixed sun gear, not to the planet cage.

6. What is the speed \( v_P \) of the planet gear center in terms of the angular velocity \( \omega_g \) of the planet cage?

Solution: \( v_P = (r_u + r_p)\omega_g \)

7. What is the angular velocity \( \omega_P \) of the planet gear in terms of the speed \( v_P \) of the planet gear center?

Hint: what is the velocity of point A?

Solution: \( \omega_P = \frac{v_P}{r_p} \)
8. What is the speed $v_Q$ of the ring gear in terms of $v_p$ and $\omega_p$?

**Solution:** $v_Q = v_p + r_p \omega_p$ or $v_Q = 2 \omega_p r_p = 2v_p$

9. What is the angular velocity $\omega_r$ of the ring gear in terms of the speed $v_Q$?

**Solution:** $\omega_r = \frac{v_Q}{r_r}$

10. Using your answers above, derive an expression for $\omega_r$ in terms of $\omega_g$ and the radii $r_u$ and $r_p$. Observe that $r_r = r_u + 2r_p$.

**Solution:**

\[
\omega_r = \frac{v_Q}{r_r} = \frac{v_p + r_p \omega_p}{r_u + 2r_p} = \frac{2v_p}{r_u + 2r_p} = \frac{2r_u + 2r_p}{r_u + 2r_p} \omega_g
\]

11. The Shimano Nexus hub on Prof. West’s bike has the radius of the sun and planet gears related by $r_p = 0.88r_u$. What is the ratio $\omega_r/\omega_g$?

**Solution:**

\[
\frac{\omega_r}{\omega_g} = \frac{2r_u + 2r_p}{r_u + 2r_p} = \frac{2r_u + 2 \times 0.88r_u}{r_u + 2 \times 0.88r_u} = \frac{2 + 2 \times 0.88}{1 + 2 \times 0.88} \approx 1.36
\]

12. The Shimano Nexus hub gear can operate in three different configurations, selected by the gear shifter:

<table>
<thead>
<tr>
<th>Gear</th>
<th>Rear wheel connection</th>
<th>Sprocket connection</th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>planet cage ($\omega_w = \omega_g$)</td>
<td>ring gear ($\omega_s = \omega_r$)</td>
</tr>
<tr>
<td>G2</td>
<td>ring gear ($\omega_w = \omega_r$)</td>
<td>ring gear ($\omega_s = \omega_r$)</td>
</tr>
<tr>
<td>G3</td>
<td>ring gear ($\omega_w = \omega_r$)</td>
<td>planet cage ($\omega_s = \omega_g$)</td>
</tr>
</tbody>
</table>

What is the ratio $\omega_w/\omega_s$ in each of the three gears?

**Solution:**

\[
G1: \quad \frac{\omega_w}{\omega_s} = \frac{\omega_g}{\omega_r} = \frac{1}{1.36} \approx 0.73
\]
\[
G2: \quad \frac{\omega_w}{\omega_s} = \frac{\omega_r}{\omega_r} = 1
\]
\[
G3: \quad \frac{\omega_w}{\omega_s} = \frac{\omega_r}{\omega_g} = 1.36
\]
13. While riding into the office, Prof. West observes that he is pedaling at $f_C = 1$ Hz (so $\omega_C = 2\pi f_C \approx 6.28$ Hz). Prof. West’s bike also uses 700C wheels (so $r_w \approx 0.35$ m), has a chainring size of $r_C = 0.085$ m, and has a sprocket size of $r_S = 0.034$ m. What speeds, $v_w$, could be expected for the G1 and G3 ratios? How does it compare with Prof. Ertekin’s? (from Q5)

Hint: Use the ratio $\omega_w/\omega_s$ from Q12, and the expression for the ratio $\omega_s/\omega_c$ from Q3. Discuss the difference between the answers from Q5 and Q13.

Solution:

\[
\begin{align*}
G1 &: v_w \approx 14.4\text{km/h} = 4\text{m/s} \\
G3 &: v_w \approx 26.9\text{km/h} = 7.47\text{m/s}
\end{align*}
\]

The range of speed is much narrower in this case.

**Bonus Questions**

14. Derailleur gears are the most popular transmission for high performance road and mountain bikes, whereas for urban commuter bikes hub gears are increasingly popular. What are pros and cons of each gear type that lead to use in these different applications?

Hint: consider friction, reliability, repairability, maintenance, ease of shifting, etc.

**Solution:** Derailleur gears have lower friction and are much easier to access for maintenance and repairs. Hub gears, however, are sealed and so less subject to dirt and water, reducing maintenance requirements. Hub gears can be shifted while stationary, and require less operator training to use.

15. Normally chainring and sprocket sizes are not listed in terms of the physical radius, but rather are given in terms of the number of teeth on them. Why is this sufficient to determine gear ratios?

**Solution:** All that matters is the ratio of gear radii, or equivalently the ratio of gear circumferences. This is given by the number of teeth multiplied by the pitch of the chain (the tooth spacing), so the tooth count ratio is equal to the circumference ratio and thus also the diameter ratio.