TAM 212 Worksheet 7: Car steering

This worksheet aims to understand how cars steer. The #avs webpage on “Steering geometry” illustrates the basic ideas. On the diagram below, the \textit{kingpins} at \( A \) and \( B \) are a distance \( g \) apart (this is almost the same as the \textit{track} distance between the front wheel tire centers), while the \textit{wheelbase} distance is \( AF = BE = \ell \). Modern cars use ball joints instead of actual pins at the kingpin joints.

1. Consider the four-wheeled car configuration shown above. The left-front wheel is turned at an angle of \( \theta_L \), and the turning radius of the car is \( \rho \), measured from the center \( P \) of the rear axle to the instantaneous center \( M \). Derive a formula for \( \rho \) in terms of \( \theta_L \), leaving measurements \( g \) and \( \ell \) in symbolic form.

\textbf{Solution: } We see \( \angle AMF = \theta_L \), so \( \tan \theta_L = AF/MF = \ell/(\rho - g/2) \), giving:

\[
\rho = \frac{\ell}{\tan \theta_L} + \frac{g}{2}
\]

2. Similarly to the previous question, derive a formula for \( \rho \) in terms of the angle \( \theta_R \) of the right-front wheel.

\textbf{Solution: } Similar to the previous question:

\[
\rho = \frac{\ell}{\tan \theta_R} - \frac{g}{2}
\]
3. While trying to park our car in a tight spot, we want to drive our car around a counter-clockwise curve with a radius of curvature of $\rho = 5$ m. At what angles $\theta_L$ and $\theta_R$ should we ideally set our wheels, in order to make this turn? Give your answers in numeric form.

**Solution:** Solving the Q1 and Q2 equations:

$$\theta_L = \tan^{-1}(3/4) \approx 36.9^\circ$$

$$\theta_R = \tan^{-1}(1/2) \approx 26.6^\circ$$

4. Ackermann steering geometry, shown in the figure below, uses a four-bar linkage $ABCD$ to constrain the wheel angles $\theta_L$ and $\theta_R$. The tie rod has length $CD = f$, while the steering arms have lengths $AD = a$ and $BC = b$. A simple rule of thumb for designing Ackermann steering sets the linkage geometry so that the steering arms point to the center $P$ of the rear axle, as shown. Given lengths $a = b = 0.2$ m, what is the angle $\gamma$ and the appropriate length $f$ of the tie rod?

**Solution:**

$$\tan \gamma = \frac{g/2}{\ell}$$

$$\gamma \approx 18.4^\circ$$

$$\sin \gamma = \frac{(g-f)/2}{a}$$

$$f = g - 2a \sin \gamma$$

$$\approx 1.87 \text{ m}$$
5. The initial and turned state of front wheels of Ackermann steering geometry case are drawn as below. On this figure, indicate the turning angles of right and left wheel ($\theta^*_R$ and $\theta^*_L$), and $\gamma$.

Solution:
6. $\theta_L$ and $\theta_R$ we found from Q3 are ideal angles during the turn without the tie rod $CD$. We want to see how well the Ackermann steering geometry we designed in the Q4 works. Consider the turn from Q3 with $\rho = 5$ m, and set the left-front wheel angle $\theta^*_L$ is equal to the value $\theta_L$ found in Q3 ($\theta^*_L = \theta_L$). Also, the geometric results from Q5 should be helpful.

What right-front angle $\theta^*_R$ is now determined by the linkage? Use the diagram below to start with $\theta_L$ and work your way across the diagram to find $\theta^*_R$. The law of cosines will be helpful for determine angles on general triangles, for example

$$c^2 = a^2 + g^2 - 2ag \cos \angle DAB$$

<table>
<thead>
<tr>
<th>Figure</th>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\triangle ADB$</td>
<td>$\angle DAB$</td>
<td>$\angle DAB = 90^\circ - \gamma - \theta_L$</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>$c \approx 1.84$ m</td>
</tr>
<tr>
<td>$\triangle ADB$</td>
<td>$\angle ABD$</td>
<td></td>
</tr>
<tr>
<td>$\triangle DBC$</td>
<td>$\angle DBC$</td>
<td></td>
</tr>
<tr>
<td>$\square ABCD$</td>
<td>$\angle ABC$</td>
<td>$\angle ABC = \angle ABD + \angle DBC \approx 100.4^\circ$</td>
</tr>
<tr>
<td></td>
<td>$\theta^*_R$</td>
<td>$\theta^*_R = \angle ABC + \gamma - 90^\circ \approx 28.8^\circ$</td>
</tr>
</tbody>
</table>

**Solution:** Using the law of cosines:

$$\angle DAB = 90^\circ - \gamma - \theta_L$$

$$= 34.7^\circ$$

$$c^2 = a^2 + g^2 - 2ag \cos \angle DAB$$

$$c \approx 1.84 \text{ m}$$

$$\cos \angle ABD = \frac{g^2 + c^2 - a^2}{2gc}$$

$$\angle ABD \approx 3.55^\circ$$

$$\cos \angle DBC = \frac{c^2 + b^2 - f^2}{2bc}$$

$$\angle DBC \approx 96.9^\circ$$

$$\angle ABC = \angle ABD + \angle DBC$$

$$\approx 100.4^\circ$$

$$\angle ABC = 90^\circ - \gamma + \theta^*_R$$

$$\theta^*_R = \angle ABC + \gamma - 90^\circ$$

$$\approx 28.8^\circ$$
7. How close is the Ackermann value of $\theta^*_R$ from Q6 to the ideal value $\theta_R$ from Q3? Is this Ackermann steering geometry acceptable for real-world usage?

**Solution:** The ideal value $\theta_R$ and the Ackermann angle $\theta^*_R$ are:

$$\begin{align*}
\theta_R &\approx 26.6^\circ \\
\theta^*_R &\approx 28.8^\circ \\
\theta^*_R - \theta_R &\approx 2.2^\circ
\end{align*}$$

This is a small enough difference that the Ackermann steering system would perform very close to ideal in practical applications, even on such sharp turns as $\rho = 5$ m. On more gradual turns the difference will be even smaller.
Bonus Questions

8. While making the turn in the above question, we measure the speed of point $P$ to be $v_P = 2 \text{ m/s}$ (we are parking very quickly!). What is the angular velocity $\omega$ of the car during the turn?

Solution:

$$\omega = \frac{v}{\rho} = 0.4 \text{ \text{rad/s}}$$

9. While turning as above, what are the speeds of the four wheel joints $v_A$, $v_B$, $v_E$, and $v_F$?

Solution: The distances from the instantaneous center $M$ are:

$$r_A = \sqrt{(\rho - g/2)^2 + \ell^2} = 5 \text{ m}$$
$$r_B = \sqrt{(\rho + g/2)^2 + \ell^2} \approx 6.71 \text{ m}$$
$$r_E = \rho + g/2 = 6 \text{ m}$$
$$r_F = \rho - g/2 = 4 \text{ m}.$$

The speeds are thus:

$$v_A = r_A \omega = 2 \text{ m/s}$$
$$v_B = r_B \omega \approx 2.68 \text{ m/s}$$
$$v_E = r_E \omega = 2.4 \text{ m/s}$$
$$v_F = r_F \omega = 1.6 \text{ m/s}.$$

10. Considering the velocities of the four wheel joints you found in Q9, would it be reasonable to build a rear-wheel-drive car with a rear driveshaft consisting of a single rod bolted to each wheel? Why or why not? What might an alternative be?

Solution: It would not be reasonable to use a solid rear driveshaft bolted to both wheels, as the outer wheel (rear right) is moving 50% faster than the inner wheel (rear left), and so must also be rotating 50% faster. If a single driveshaft was used then one or both wheels would be forced to slip. Instead cars use a differential to allow the rear wheels to turn at different rates.