Recap
• Center of Mass, Moments of Inertia

Today
• Moments of Inertia, Kinetics of Rigid Bodies

Point mass

\[ I_{OZ} = mr^2 \]

Simple shapes: tables

Disk:
\[ I_{CZ} = \frac{1}{2}mr^2 \]

Arbitrary rigid body

\[ I_{CZ} = \iiint_B \rho r^2 \, dV \]

Parallel Axis Theorem
\[ I_{PZ} = I_{CZ} + mr_{CP}^2 \]

must be from COM

Additive Theorem
\[ I_{\text{total}}^{PZ} = I_{PZ}^{B_1} + I_{PZ}^{B_2} \]

must be on the same point, along the same axis

First use Parallel Axis Theorem, then Additive Theorem

\[ r_{CP} = \text{distance orthogonal to Z-axis} \]
Rigid Body Kinetics

Kinematics $\rightarrow$ $\vec{r}, \vec{v}, \vec{a}, \vec{\omega}, \vec{\alpha}$ (no forces)

Kinetics $\rightarrow$ $\sum \vec{F} = m \vec{a}$, $\sum \vec{M} = I \vec{\alpha}$, forces, moments

Particle: $\sum \vec{F} = m \vec{a}$

Rigid Body:

$\sum \vec{F} = m \vec{a}_C$

$\sum M_{CZ} = I_{CZ} \alpha_Z$  \hspace{1cm} C = CoM = Center of Mass

$\sum M_{OZ} \vec{r} = I_{OZ} \alpha_Z$  \hspace{1cm} O = Fixed Point

Forces & Moments $\leftarrow \text{inertia} \rightarrow$ Motion $\vec{a}, \vec{\alpha}$

$m, I$
Example

Disk starts at rest.
No gravity.
Force $D_x > 0$ applied at P

Initially

- **P accelerates**
  - A. left
  - B. zero
  - C. right

- **C accelerates**
  - A. left
  - B. zero
  - C. right

- **Q accelerates**
  - A. left
  - B. zero
  - C. right
Example

Disk starts at rest.
No gravity.
Force $D_x > 0$ applied at P

\[ a_{Cx} = \begin{align*}
A. & \quad \frac{2D_x}{m} \\
B. & \quad \frac{D_x}{m} \\
C. & \quad 0 \\
D. & \quad \frac{-D_x}{m} \\
E. & \quad \frac{-2D_x}{m}
\end{align*} \]

\[ \alpha_z = \begin{align*}
A. & \quad \frac{2D_x}{mr} \\
B. & \quad \frac{D_x}{mr} \\
C. & \quad 0 \\
D. & \quad \frac{-D_x}{mr} \\
E. & \quad \frac{-2D_x}{mr}
\end{align*} \]

\[ a_{Qx} = \begin{align*}
A. & \quad \frac{2D_x}{m} \\
B. & \quad \frac{D_x}{m} \\
C. & \quad 0 \\
D. & \quad \frac{-D_x}{m} \\
E. & \quad \frac{-2D_x}{m}
\end{align*} \]
Moments

\[ \vec{M}_C = \vec{r}_{CP} \times \vec{F}_P \]

moment about C due to force at P
Example: Rigid-rod Pendulum

1. System diagram

Rigid rod, length $\ell$, mass $m$, gravity $g$

Find: angular acceleration as a function of $\theta$, $\dot{\theta}$

2. FBDs
Example: Rigid-rod Pendulum

3. Kinematics: $\ddot{a}_C, \ddot{a}$
3. Kinematics: $\vec{a}_C, \vec{\alpha}$

$\alpha_z = A. \ r\ddot{\theta}$

B. $-r\dot{\theta}^2$

C. $\ddot{r}$

D. $2r\dot{\theta}^2$

E. $\ddot{\theta}$

Coordinates for $\vec{a}_C$?

A. Cartesian

B. Polar
Example: Rigid-rod Pendulum

4. Kinetics (Newton):

\[ \vec{R} = R_x \hat{i} + R_y \hat{j} = R_r \hat{r} + R_\theta \hat{\theta} \]

\[ M_{CZ} = \begin{cases} \frac{\ell}{2} R_r \\ \frac{\ell}{2} R_\theta \\ -\frac{\ell}{2} R_r \\ -\frac{\ell}{2} R_\theta \\ \text{NOTA} \end{cases} \]
Example: Rigid-rod Pendulum

4. Kinetics (Newton):
\[
\ddot{R} = R_r \hat{e}_r + R_\theta \hat{e}_\theta \\
\ddot{a} = (\ddot{r} - r \dot{\theta}^2) \hat{e}_r + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \hat{e}_\theta
\]
\[
\Sigma \vec{F} = m \ddot{a}_C \Rightarrow \ddot{R} - mg \hat{j} = -m \frac{\ell}{2} \dot{\theta}^2 \hat{e}_r + m \frac{\ell}{2} \ddot{\theta} \hat{e}_\theta \tag{1}
\]
\[
\Sigma M_{CZ} = I_{CZ} \alpha_z \quad I_{CZ} = \frac{1}{12} ml^2
\]
\[
\vec{M}_C = \vec{r}_{CO} \times \ddot{R} = \left(-\frac{\ell}{2} \hat{e}_r\right) \times (R_r \hat{e}_r + R_\theta \hat{e}_\theta) = -\frac{\ell}{2} R_\theta \hat{k}
\]
\[
M_{CZ} = I_{CZ} \alpha_z \Rightarrow -\frac{\ell}{2} R_\theta = \frac{1}{12} ml^2 \ddot{\theta} \tag{2}
\]

5. Algebra

\(1\) \(\theta\)-direction \(\Rightarrow\) \(R_\theta - mg \sin \theta = m \frac{\ell}{2} \ddot{\theta}\) \(\tag{3}\)

\(2\) \(\Rightarrow\) \(-\ell R_\theta = \frac{1}{6} ml^2 \ddot{\theta}\) \(\tag{4}\)

\(\ell (3 + 4) \Rightarrow -mg \ell \sin \theta = \frac{2}{3} ml^2 \ddot{\theta}\)

\[\ddot{\theta} = -\frac{3g}{2\ell} \sin \theta\] Solve this numerically
Example: Rigid-rod Pendulum
Alternative solution using fixed-point O

4. Kinetics (Newton): \[ \ddot{a} = \left( \ddot{r} - r \dot{\theta}^2 \right) \hat{e}_r + \left( r \ddot{\theta} + 2 \dot{r} \dot{\theta} \right) \hat{e}_\theta \]

\[ \Sigma M_{OZ} = I_{OZ} \alpha_Z \quad I_{OZ} = \frac{1}{3} m \ell^2 \]

\[ M_O = \vec{r}_{Oc} \times \vec{F}_g = \left( \frac{\ell}{2} \hat{\theta} \right) \times (mg \cos \theta \hat{e}_r - mg \sin \theta \hat{e}_\theta) \]

\[ M_O = -mg \frac{\ell}{2} \sin \theta \hat{k} \]

\[ M_{OZ} = I_{OZ} \alpha_Z \Rightarrow -mg \frac{\ell}{2} \sin \theta = \frac{1}{3} m \ell^2 \ddot{\theta} \]

5. Algebra

\[ \ddot{\theta} = -\frac{3}{2} \frac{g}{\ell} \sin \theta \]
Instantaneous Centers (IC)

Rigid body moving in 2D.
• Rotating and possible translating

IC = M = point that is not moving
    = point that RB is rotating about
    (only defined instantaneously)
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Which point is M?
Instantaneous Centers (IC)

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Which point is M?
IC Examples

Which point is M?

Direction of $\vec{v}_B$?
IC Examples

Which point is M?

Direction of $\vec{v}_B$?

Which point is M?

Direction of $\vec{v}_B$?
Which point is M?
Intuition for IC

From M, \( \vec{v}_P \) is orthogonal to \( \vec{r}_{MP} \) (why?)

\[ \vec{v}_P = \omega \vec{r}_{MP} \]

speed proportional to distance from M
More IC Examples

Which point is M?
More IC Examples

Which point is M?

Which point is M?
Even More IC Examples

Which point is M?

A B C D E - none

Which point is M?

A B C D E - none
Graphical Rules for Finding M (the IC)

Draw lines perpendicular to velocities
- If the lines intersect at a single point
  ⇒ that point is M
- If the lines are the same lines
  ⇒ Draw a line through the velocity tips
- If the lines intersect at a single point
  ⇒ that point is M

Careful!
- Consistent direction of rotation
- Consistent speeds $v = \omega r$
- Body may not be rotating (pure translation)
Which body is translating?

A. RB #1
B. RB #2
C. both
D. neither
E. can’t tell
Some Key Points

- Rotation is a property of the body as a whole
- Translation really isn’t such a property (unless we have pure translation)
- We can talk about translation of specific points, like center C
Calculating the Position of M

Key property: M is the point with \( \vec{v}_M = 0 \) at that instant OR \( \vec{v}_P \) is in pure rotation about M.

Example

\[ \vec{v}_C = (4, 2) \text{m/s} \]
\[ \vec{\omega} = -2 \text{ rad/s} \]

Where is M relative to C?
Example

\[ \vec{r}_A = (1, 4) \text{m} \quad \vec{v}_A = (-3, -9) \text{m/s} \]
\[ \vec{r}_B = (3, 3) \text{m} \quad \vec{v}_B = (0, -3) \text{m/s} \]

Where is M?
Example

The Ladder is sliding down the wall.