The aim of this worksheet is to understand how vehicles drive around curves, how slipping and rolling limit the maximum speed, and how banking the road can increase the maximum speed.

The photos above show two different views of a Setra S 411 HD bus cornering at high speed on the Mercedes-Benz test track at Untertürkheim, on the Neckar River just outside of Stuttgart, Germany.

Below is a top view of the track, showing the bus and its velocity and acceleration vectors. In this worksheet we will always assume a circular track of fixed radius $\rho$ and a bus traveling at constant speed $v.$
Cornering on a flat road

First we consider the bus driving around a curve on a flat horizontal road, as shown in the front-view free body diagram below. We will model the bus as a rigid body that contacts the ground at two points (the wheel centers) with normal forces $N_1$ and $N_2$, and we assume that the friction force $F$ applies equally to each wheel. Gravity $g$ acts through the center of mass $C$.

1. What is the acceleration $\ddot{a}$ as a function of the bus speed $v$ and radius of curvature $\rho$ of the road?

Solution: $\ddot{a} = -\frac{v^2}{\rho}\hat{i}$
2. If the bus is not rolling then it has zero angular velocity about the \( \hat{k} \) direction, so the total moment about the \( \hat{k} \) axis will be zero. The kinetics will thus satisfy \( \sum \vec{F} = ma \) and \( \sum \vec{M}_C = 0 \). Use these two equations to find the forces \( F, N_1, \) and \( N_2 \) in terms of all other variables.

Solution: The rigid body equations are:

\[
\sum \vec{F} = ma \\
-F_i + N_1j + N_2j - mgj = ma = -\frac{mv^2}{\rho}i \\
\sum \vec{M}_C = 0 \\
-Fh\hat{k} - N_1\ell\hat{k} + N_2\ell\hat{k} = 0.
\]

Equating components gives the equations:

\[
-F = -\frac{mv^2}{\rho} \\
N_1 + N_2 - mg = 0 \\
-Fh - N_1\ell + N_2\ell = 0
\]

and solving these gives:

\[
F = \frac{mv^2}{\rho} \\
N_1 = \frac{mg}{2} - \frac{mv^2h}{2\rho\ell} \\
N_2 = \frac{mg}{2} + \frac{mv^2h}{2\rho\ell}
\]

3. The bus will not slide if \( F \leq F_{max} = \mu(N_1 + N_2) \), where \( \mu \) is the coefficient of friction for the tires on the road. For a track radius of \( \rho = 50 \) m and a coefficient of friction \( \mu = 0.3 \), what is the fastest that the bus can drive around the corner before it starts to slide?

Solution: Critical speed occurs for:

\[
F = F_{max} = \mu(N_1 + N_2) = \mu mg \\
\frac{mv^2}{\rho} = \mu mg \\
v = \sqrt{\mu g \rho} \\
\approx 12.1 \text{ m/s} \approx 43 \text{ km/h}
\]
4. The bus will not roll if $N_1 \geq 0$. For a track radius of $\rho = 50$ m and bus geometry $\ell = 0.97$ m and $h = 0.87$ m. What is the fastest the bus can drive around the corner before it rolls over?

**Solution:** Critical speed occurs for:

\[
0 = N_1 = \frac{mg}{2} - \frac{mv^2h}{2\rho\ell}
\]

\[
\frac{mv^2h}{2\rho\ell} = \frac{mg}{2}
\]

\[
v = \sqrt{\frac{g\rho\ell}{h}}
\]

\[
\approx 23.4 \text{ m/s} \approx 84.2 \text{ km/h}
\]

5. If the bus tries to drive at high speed around a track as described in the last two questions, will sliding or rolling be the limiting factor? What is the maximum speed at which the bus can drive around this curve?

**Solution:** Sliding will occur at a lower speed, so it is the limiting factor and the maximum speed is $v \approx 12.1$ m/s $\approx 43$ km/h.

6. Would altering the dimensions $\ell$ and $h$ of the bus change the speed at which it slides? Would it change the speed at which it rolls? How should $\ell$ and $h$ be changed to make the bus harder to roll?

**Solution:** From the answer to Question 3 we see that the sliding speed is independent of $\ell$ and $h$. However, from the answer to Question 4 we see that the speed when rolling occurs does depend on $\ell$ and $h$. To make this speed higher (so harder to roll) we should make $\ell$ larger and $h$ smaller, as we see from the Question 4 solution. This would tend to make the bus wide and low, like a tank.
Cornering on a banked turn

We now consider the situation when the road is banked, as shown in the free body diagram below.

7. The safest driving conditions are when there is no need for a friction force, so $F = 0$. Write Newton’s equations in the $\hat{I}$ direction in this case and obtain a formula for the safest speed $v$ to drive for a given bank angle $\theta$.

Solution:

$$\sum F_1 = ma_1$$

$$-mg \sin \theta = -ma \cos \theta$$

$$-mg \sin \theta = -\frac{mv^2}{\rho} \cos \theta$$

$$v = \sqrt{\rho g \tan \theta}$$
8. For the bus shown on page 1, we have $\theta = 45^\circ$ and $\rho = 50$ m. Assuming the bus is driving at the safest speed (so $F = 0$), what is its speed?

**Solution:**

$$v = \sqrt{\rho g \tan \theta}$$

$$\approx 22.1 \text{ m/s} \approx 79.7 \text{ km/h}$$

9. If we wanted to drive the bus at $v = 40$ m/s around the banked track with radius $\rho = 50$ m, what bank angle would be necessary (assuming we have the safest conditions of $F = 0$)?

**Solution:**

$$v = \sqrt{\rho g \tan \theta}$$

$$\tan \theta = \frac{v^2}{\rho g}$$

$$\theta \approx 73.0^\circ$$

As a matter of interest, 40 m/s $\approx 144$ km/h.

10. Is there any limit to how fast the bus can drive around the banked track, given that we can make the bank angle as close to $\theta = 90^\circ$ as we like?

**Solution:** Our rigid body model does not show any limit to the speed, as we can just keep increasing $\theta$ to ensure that $F = 0$. In practice the normal acceleration will be so high that the people on the bus will start to black out. Before this point the driver of the bus will probably lose control, due to the high speed and small margin for error at high bank angles.
Challenge questions

11. For the banked curve, let us consider the general case where $\vec{F} \neq 0$. Use $\sum \vec{F} = ma$ and $\sum \vec{M}_C = 0$ to derive expressions for $F$, $N_1$, and $N_2$ in terms of the other variables.
   Hint: use the $\hat{I}, \hat{J}$ basis.

Solution:

\[ \sum F_I = ma_I \]
\[ -F - mg\sin \theta = -\frac{mv^2}{\rho} \cos \theta \]

\[ \sum F_J = ma_J \]
\[ N_1 + N_2 - mg \cos \theta = \frac{mv^2}{\rho} \sin \theta \]

\[ \sum \vec{M}_C = 0 \]
\[ -Fh - N_1 \ell + N_2 \ell = 0 \]

Equating components gives the equations:

\[ F = \frac{mv^2}{\rho} \cos \theta - mg \sin \theta \]
\[ N_1 + N_2 = mg \cos \theta - \frac{mv^2}{\rho} \sin \theta \]
\[ N_1 - N_2 = -\frac{Fh}{\ell} = -\frac{h}{\ell} \left( \frac{mv^2}{\rho} \cos \theta - mg \sin \theta \right) \]

and solving these gives:

\[ \therefore F = \frac{mv^2}{\rho} \cos \theta - mg \sin \theta \]
\[ \therefore N_1 = \frac{m}{2} \left\{ g(\cos \theta + \frac{h}{\ell} \sin \theta) + \frac{v^2}{\rho} (\sin \theta - \frac{h}{\ell} \cos \theta) \right\} \]
\[ \therefore N_2 = \frac{m}{2} \left\{ g(\cos \theta - \frac{h}{\ell} \sin \theta) + \frac{v^2}{\rho} (\sin \theta + \frac{h}{\ell} \cos \theta) \right\} \]

12. Using your answer to Question 11 and the same sliding restriction as in Question 3, how fast can the bus drive around a track with bank angle $\theta = 30^\circ$ before it starts to slide?
Solution:

\[ F \leq F_{\text{max}} = \mu(N_1 + N_2) \]

\[ \frac{mv^2}{\rho} \cos \theta - mg \sin \theta \leq \mu(mg \cos \theta + \frac{mv^2}{\rho} \sin \theta) \]

\[ v \leq \sqrt{\rho g \frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta}} \]

\[ \therefore v \approx 22.81 \text{ (m/s)} \approx 46.19 \text{ (km/h)} \]

13. Using your answer to Question 11 and the same rolling restriction as in Question 4, how fast can the bus drive around a track with bank angle \( \theta = 30^\circ \) before it starts to roll?

Solution:

\[ N_1 \geq 0 \]

\[ N_1 = \frac{m}{2} \left\{ g(\cos \theta + \frac{h}{l} \sin \theta) + \frac{v^2}{\rho} (\sin \theta - \frac{h}{l} \cos \theta) \right\} \geq 0 \]

\[ v = \sqrt{\rho g \frac{\cos \theta + \frac{h}{l} \sin \theta}{\cos \theta - \frac{h}{l} \sin \theta}} \]

\[ \therefore v \approx 48.27 \text{ (m/s)} \approx 173.8 \text{ (km/h)} \]

14. If the bus tries to drive at high speed around a track with bank angle \( \theta = 30^\circ \), as described in the last two questions, will sliding or rolling be the limiting factor? What is the maximum speed at which the bus can drive around this curve?

Solution:

*Sliding is a limiting factor.*