TAM 212 Worksheet 7: Car steering

This worksheet aims to understand how cars steer. The #avs webpage on “Steering geometry” illustrates the basic ideas. On the diagram below, the kingpins at A and B are a distance $g$ apart (this is almost the same as the track distance between the front wheel tire centers), while the wheelbase distance is $AF = BE = \ell$. Modern cars use ball joints instead of actual pins at the kingpin joints.

1. Consider the four-wheeled car configuration shown above. The left-front wheel is turned at an angle of $\theta_L$, and the turning radius of the car is $\rho$, measured from the center $P$ of the rear axle to the instantaneous center $M$. Derive a formula for $\rho$ in terms of $\theta_L$, leaving measurements $g$ and $\ell$ in symbolic form.

**Solution:** We see $\angle AMF = \theta_L$, so $\tan \theta_L = AF/MF = \ell/(\rho - g/2)$, giving:

$$\rho = \frac{\ell}{\tan \theta_L} + \frac{g}{2}$$

2. Similarly to the previous question, derive a formula for $\rho$ in terms of the angle $\theta_R$ of the right-front wheel.

**Solution:** Similar to the previous question:

$$\rho = \frac{\ell}{\tan \theta_R} - \frac{g}{2}$$
3. While trying to park our car in a tight spot, we want to drive our car around a counter-clockwise curve with a radius of curvature of $\rho = 5$ m. At what angles $\theta_L$ and $\theta_R$ should we ideally set our wheels, in order to make this turn? Give your answers in numeric form.

**Solution:** *Solving the Q1 and Q2 equations:*

\[ \theta_L = \tan^{-1}(3/4) \approx 36.9^\circ \]
\[ \theta_R = \tan^{-1}(1/2) \approx 26.6^\circ \]

4. Ackermann steering geometry, shown in the figure below, uses a four-bar linkage $ABCD$ to constrain the wheel angles $\theta_L$ and $\theta_R$. The tie rod has length $CD = f$, while the steering arms have lengths $AD = a$ and $BC = b$. A simple rule of thumb for designing Ackermann steering sets the linkage geometry so that the steering arms point to the center $P$ of the rear axle, as shown. Given lengths $a = b = 0.2$ m, what is the angle $\gamma$ and the appropriate length $f$ of the tie rod?

**Solution:**

\[ \tan \gamma = \frac{g/2}{\ell} \]
\[ \gamma \approx 18.4^\circ \]
\[ \sin \gamma = \frac{(g - f)/2}{a} \]
\[ f = g - 2a \sin \gamma \]
\[ \approx 1.87 \text{ m} \]
5. The initial and turned state of front wheels of Ackermann steering geometry case are drawn as below. On this figure, indicate the turning angles of right and left wheel ($\theta^*_R$ and $\theta^*_L$), and $\gamma$.

Solution:

![Diagram of Ackermann steering geometry](image)

6. The angles $\theta_L$ and $\theta_R$ we found from Q3 are ideal angles during the turn without the tie rod $CD$. We want to see how well the Ackermann steering geometry we designed in the Q4 works. Consider the turn from Q3 with $\rho = 5$ m, and set the left-front wheel angle $\theta^*_L$ is equal to the value $\theta_L$ found in Q3 ($\theta^*_L = \theta_L$). Also, the geometric results from Q5 should be helpful.

What right-front angle $\theta^*_R$ is now determined by the linkage? Use the diagram below (which is a simplification of the linkage diagram above) to start with $\theta_L$ and work your way across the diagram to find $\theta^*_R$. The law of cosines will be helpful for determine angles on general triangles, for example

$$c^2 = a^2 + g^2 - 2ag \cos \angle DAB$$

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<thead>
<tr>
<th>Figure</th>
<th>Quantity</th>
<th>Value</th>
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<tbody>
<tr>
<td>$\triangle ADB$</td>
<td>$\angle DAB$</td>
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<td></td>
<td>$c$</td>
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<td>$\triangle ADB$</td>
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<td>$\triangle DBC$</td>
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<td>$\square ABCD$</td>
<td>$\angle ABC$</td>
<td>$\angle ABC = \angle ABD + \angle DBC =$</td>
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<td></td>
<td>$\theta^*_R$</td>
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Solution: Using the law of cosines:

\[
\angle DAB = 90^\circ - \gamma - \theta_L
\]

\(= 34.7^\circ\)

\[c^2 = a^2 + g^2 - 2ag \cos \angle DAB\]

\(c \approx 1.84 \text{ m}\)

\[\cos \angle ABD = \frac{g^2 + c^2 - a^2}{2gc}\]

\(\angle ABD \approx 3.55^\circ\)

\[\cos \angle DBC = \frac{c^2 + b^2 - f^2}{2cb}\]

\(\angle DBC \approx 96.9^\circ\)

\(\angle ABC = \angle ABD + \angle DBC\)

\(\approx 100.4^\circ\)

\(\angle ABC = 90^\circ - \gamma + \theta^*_R\)

\(\theta^*_R = \angle ABC + \gamma - 90^\circ\)

\(\approx 28.8^\circ\)

7. How close is the Ackermann value of \(\theta^*_R\) from Q6 to the ideal value \(\theta_R\) from Q3? Is this Ackermann steering geometry acceptable for real-world usage?

Solution: The ideal value \(\theta_R\) and the Ackermann angle \(\theta^*_R\) are:

\(\theta_R \approx 26.6^\circ\)

\(\theta^*_R \approx 28.8^\circ\)

\(\theta^*_R - \theta_R \approx 2.2^\circ\)

This is a small enough difference that the Ackermann steering system would perform very close to ideal in practical applications, even on such sharp turns as \(\rho = 5 \text{ m}\). On more gradual turns the difference will be even smaller.
Challenge Questions

8. While making the turn in the above question, we measure the speed of point $P$ to be $v_P = 2 \text{ m/s}$ (we are parking very quickly!). What is the angular velocity $\omega$ of the car during the turn?

Solution:

$$\omega = \frac{v}{\rho} = 0.4 \text{ rad/s}$$

9. While turning as above, what are the speeds of the four wheel joints $v_A$, $v_B$, $v_E$, and $v_F$?

Solution: The distances from the instantaneous center $M$ are:

$$r_A = \sqrt{\left(\rho - g/2\right)^2 + \ell^2} = 5 \text{ m}$$
$$r_B = \sqrt{\left(\rho + g/2\right)^2 + \ell^2} \approx 6.71 \text{ m}$$
$$r_E = \rho + g/2 = 6 \text{ m}$$
$$r_F = \rho - g/2 = 4 \text{ m}.$$ 

The speeds are thus:

$$v_A = r_A \omega = 2 \text{ m/s}$$
$$v_B = r_B \omega \approx 2.68 \text{ m/s}$$
$$v_E = r_E \omega = 2.4 \text{ m/s}$$
$$v_F = r_F \omega = 1.6 \text{ m/s}.$$ 

10. Considering the velocities of the four wheel joints you found in Q9, would it be reasonable to build a rear-wheel-drive car with a rear driveshaft consisting of a single rod bolted to each wheel? Why or why not? What might an alternative be?

Solution: It would not be reasonable to use a solid rear driveshaft bolted to both wheels, as the outer wheel (rear right) is moving 50% faster than the inner wheel (rear left), and so must also be rotating 50% faster. If a single driveshaft was used then one or both wheels would be forced to slip. Instead cars use a differential to allow the rear wheels to turn at different rates.