TAM 212 Worksheet 3: Track design

Solutions

In this worksheet we will design one curve on the track of the Chicago O’Hare Airport Transit System (ATS):

1. On the diagram below, sketch what you think might be a good shape of the track between the two straight-line segments. Don’t spend too long on this, but just give it your best guess. In the rest of the worksheet we will calculate a precise answer.

See Question [II].

2. What were the important design considerations that you used when answering Question [II]

Smooth ride for passengers (limited acceleration); short path length.
Quarter-circle-curve design

We will first analyze a very simple design, where we just use a quarter circle for the curve:

3. If the train moves around the quarter-circle-curve track at constant speed $v = 10 \text{ m/s}$, what is its angular velocity $\omega$ about $O$ while on the circular segment?

$\omega = 0.1 \text{ rad/s}$

4. Plot the radial $a_r$ and angular $a_\theta$ components of the train’s acceleration $\vec{a} = a_r \hat{e}_r + a_\theta \hat{e}_\theta$ versus $\theta$ as the train moves around the circular segment. Clearly mark all significant points on the vertical axis.
5. People are normally standing on airport trains. As the lead engineer for the ATS system design, do you anticipate any problems for the passengers on this train as it goes along this track with a quarter-circle curve?

The radial acceleration will suddenly jump from zero on the straight-line segments to \(-1\ \text{m/s}^2\) on the curve. This will be uncomfortable for the passengers who will feel this as a sudden jerk sideways.

**Variable-radius-curve design**

6. To give passengers a smoother ride, while still keeping the train traveling at constant angular velocity \(\dot{\theta}(t) = \omega\) calculated above, we want to redesign the track so that its radial coordinate \(r_{\text{track}}(\theta)\) varies with the angle \(\theta\). Noting that \(r(t) = r_{\text{track}}(\theta)\), use the chain rule to find \(\dot{r}\) and \(\ddot{r}\) in terms of \(r'_{\text{track}}(\theta) = \frac{dr}{d\theta}\), \(r''_{\text{track}}(\theta) = \frac{d^2r}{d\theta^2}\), and \(\omega = \dot{\theta} = \frac{d\theta}{dt}\).

The chain rule gives \(\dot{r} = r'_{\text{track}} \dot{\theta}\); applying product and chain rules again gives \(\ddot{r} = r''_{\text{track}} \dot{\theta}^2 + r'_{\text{track}} \ddot{\theta}\).

7. Using the results from the previous question, write the acceleration in the polar basis in terms of \(r'_{\text{track}}(\theta)\), \(r''_{\text{track}}(\theta)\), and \(\omega\).

We note that \(\dot{\theta} = \omega\) and \(\ddot{\theta} = 0\); substitution then gives

\[
\vec{a} = (\ddot{r} - r\dot{\theta}^2) \hat{e}_r + (r\ddot{\theta} + 2r\dot{\theta} \dot{\theta}) \hat{e}_\theta = (r''_{\text{track}} - r_{\text{track}}) \omega^2 \dot{e}_r + 2r'_{\text{track}} \omega^2 \dot{\theta} \hat{e}_\theta
\]

8. From now on we will keep \(\omega\) fixed at the constant value from Question 3 and we will use the functional form:

\[
r_{\text{track}}(\theta) = A\theta^2 \left(\theta - \frac{\pi}{2}\right)^2 + 100 \text{ m}
\]

where \(\theta\) is in radians and \(A\) is an unknown constant. This function looks like:
Using this function, what are the radial and angular components of the train’s acceleration just as it enters the variable-radius curved track segment?

Evaluating the above acceleration expression with this \( r(\theta) \) gives:

\[
\begin{align*}
\mathbf{r}_{\text{track}}(\theta) &= A\theta^2 \left( \theta - \frac{\pi}{2} \right)^2 + 100 \\
\mathbf{r}'_{\text{track}}(\theta) &= 2A\theta \left( \theta - \frac{\pi}{2} \right)^2 + 2A\theta^2 \left( \theta - \frac{\pi}{2} \right) \\
\mathbf{r}''_{\text{track}}(\theta) &= 2A \left( \theta - \frac{\pi}{2} \right)^2 + 8A\theta \left( \theta - \frac{\pi}{2} \right) + 2A\theta^2 \\
\mathbf{r}_{\text{track}}(0) &= 100 \\
\mathbf{r}'_{\text{track}}(0) &= 0 \\
\mathbf{r}''_{\text{track}}(0) &= \frac{A\pi^2}{2}
\end{align*}
\]

The acceleration at \( \theta = 0 \) is thus:

\[
\mathbf{a} = \left( \frac{A\pi^2}{200} - 1 \right) \hat{e}_r \\
a_r = \frac{A\pi^2}{200} - 1 \\
a_\theta = 0
\]

9. What value of \( A \) will give an acceleration profile that does not abruptly jump when moving onto the variable-radius curve from the straight-line segment?

To have zero initial acceleration on the curve we need \( A = 200/\pi^2 \approx 20.3 \text{ m.} \)

10. Using the good value of \( A \) that you found above, what is the maximum radius of the variable-radius track?

Maximum radius is at \( \theta = \pi/4 \) which gives \( r = 107.7 \text{ m.} \)

11. Sketch the new curved track on the diagram on page 2

This is almost the same as the circular track, except it bulges outwards by 8% when \( \theta = \pi/4 \).

12. (Bonus question) What is the maximum absolute radial acceleration on the variable-radius track?

Maximum absolute radial acceleration is at \( \theta = \pi/4 \), when \( a_r = -1.57 \text{ m/s}^2 \).

13. (Bonus question) Roughly sketch the radial acceleration of the train versus \( \theta \) on the variable-radius track.
14. (Bonus question) Does the variable-radius track design have a higher or lower maximum absolute radial acceleration than the pure-quarter-circle track? In your opinion as the lead engineer, discuss the trade-offs in these two designs.

The purely circular track has radial acceleration of $-1 \text{ m/s}^2$, so the variable-radius track has 57% higher maximum radial acceleration. The new track is probably better for the passengers, however, as the maximum acceleration is still not too high, and it is easier to adapt to the acceleration if it varies smoothly.