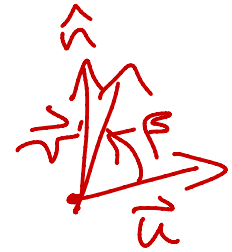
09-08-17

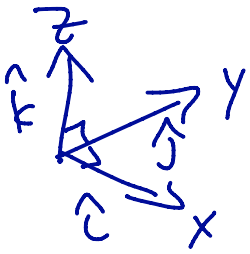
Lecture #5

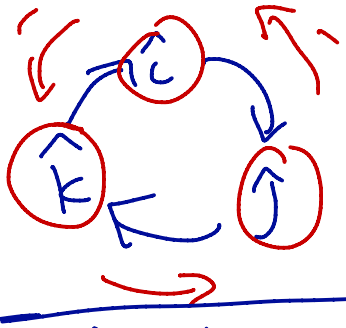
Vector products:

- dot product $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos\beta$
- cross product $\vec{u}\times\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\sin\beta\,\hat{n}$

$\|\vec{u}\| \leftarrow$ length of the vector $\vec{u}$ (euclidean)







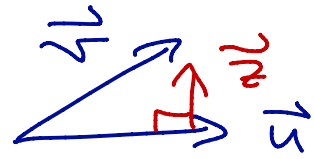
Property:

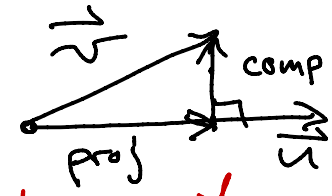
$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$
$$(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}$$

$$\vec{v}\times\vec{u} = (v_1\hat{i}+v_2\hat{j}+v_3\hat{k})\times(u_1\hat{i}+u_2\hat{j}+u_3\hat{k})$$
$$= v_1\hat{i}\times(\quad)+v_2\hat{j}\times(\quad)+v_3\hat{k}(\quad)$$
$$= v_1u_1\cancel{\hat{i}\times\hat{i}}+v_1\hat{i}\times u_2\hat{j}+v_1u_3\hat{i}\times\hat{k}+v_2u_1\hat{j}\times\hat{i}+\cancel{v_2u_2\hat{j}\times\hat{j}}$$
$$+v_2u_3\hat{j}\times\hat{k}+v_3u_1\hat{k}\times\hat{i}+v_3u_2\hat{i}\times\hat{j}+\cancel{v_3u_3\hat{k}\times\hat{k}}$$
$$= (v_2u_3-u_2v_3)\hat{i}+(v_3u_1-u_3v_1)\hat{j}+(v_1u_2-u_1v_2)\hat{k}$$

Also $\vec{v}\times\vec{u} = \det\begin{pmatrix}\hat{i} & \hat{j} & \hat{k}\\ v_1 & v_2 & v_3\\ u_1 & u_2 & u_3\end{pmatrix}$

Projections:





The $\boxed{\text{projection}}$ of a vector $\vec{v}$ onto vector $\vec{u}$ is the vector defined by

$$\text{proj}(\vec{v}, \vec{u}) = (\vec{v} \cdot \hat{u})\, \hat{u}$$

recall $\hat{u} = \frac{1}{\|u\|} \vec{u}$ when $\vec{u} \neq 0$

$$\vec{v} = \text{proj}(\vec{v}, \vec{u}) + c\,\hat{z}$$

chosen to be orthogonal to $\vec{u}$

Define the $\boxed{\text{complementary projection}}$

$$\text{comp}(\vec{v}, \vec{u}) = \vec{v} - \text{proj}(\vec{v}, \vec{u})$$

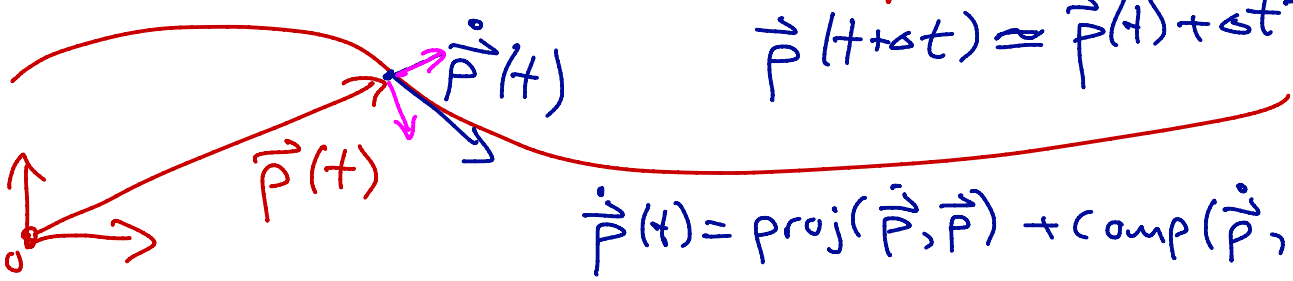
$$\Longrightarrow \vec{v} = \text{proj}(\vec{v}, \vec{u}) + \text{comp}(\vec{v}, \vec{u})$$

$$\begin{aligned}
\text{comp}(\vec{v}, \vec{u}) \cdot \vec{u} &= (\vec{v} - \text{proj}(\vec{v}, \vec{u})) \cdot \vec{u} \\
&= \vec{v} \cdot \vec{u} - \text{proj}(\vec{v}, \vec{u}) \cdot \vec{u} \\
&= \vec{v} \cdot \vec{u} - (\vec{v} \cdot \hat{u})\, \underline{\hat{u} \cdot \vec{u}} \\
&= \vec{v} \cdot \vec{u} - (\vec{v} \cdot \hat{u})\, \|u\| = \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{u} = 0
\end{aligned}$$

$\Longrightarrow$ $\text{comp}(\vec{v}, \vec{u})$ has no projection onto $\vec{u}$.

$$\text{comp}(\vec{v}, \vec{u}) \perp \text{proj}(\vec{v}, \vec{u})$$

Let's consider a time-varying position vector:



$$\vec{p}(t + \Delta t) \simeq \vec{p}(t) + \Delta t\, \dot{\vec{p}}(t)$$

$$\dot{\vec{p}}(t) = \text{proj}(\dot{\vec{p}}, \vec{p}) + \text{comp}(\dot{\vec{p}}, p)$$

Let's consider $\frac{d}{dt}\{\vec{p}(t)\} = \frac{d}{dt}\{\|\vec{p}\|\,\hat{p}(t)\}$

$$= \underline{\frac{d}{dt}\{\|\vec{p}\|\}}\hat{p}(t) + \|\vec{p}\|\,\underline{\frac{d}{dt}\{\hat{p}(t)\}}$$

Rate of change of length:

$$\frac{d}{dt}\{\|\vec{p}\|\} = \frac{d}{dt}\{(\vec{p}\cdot\vec{p})^{\frac{1}{2}}\} = \frac{1}{2}\left(\frac{1}{(\vec{p}\cdot\vec{p})^{\frac{1}{2}}}\right)\frac{d}{dt}\{\vec{p}\cdot\vec{p}\}$$

$$= \cancel{\tfrac{1}{2}}\frac{1}{\|\vec{p}\|}\underbrace{(\dot{\vec{p}}\cdot\vec{p} + \vec{p}\cdot\dot{\vec{p}})}_{=\dot{\vec{p}}\cdot\vec{p}+\dot{\vec{p}}\cdot\vec{p}} = \frac{1}{\|\vec{p}\|}\,\vec{p}\cdot\dot{\vec{p}} = \underline{\dot{\vec{p}}\cdot\hat{p}}$$

Directional change:

$$\frac{d}{dt}\{\hat{p}\} = \frac{d}{dt}\left\{\frac{1}{\|\vec{p}\|}\vec{p}\right\} = \underline{\frac{d}{dt}\left\{\frac{1}{\|\vec{p}\|}\right\}}\vec{p} + \underline{\frac{1}{\|\vec{p}\|}\dot{\vec{p}}}$$

we already computed this above

$$= -\frac{1}{\|\vec{p}\|^2}\left\{\frac{d}{dt}\{\|\vec{p}\|\}\right\}\vec{p} + \frac{1}{\|\vec{p}\|}\dot{\vec{p}} \qquad \underline{\vec{p}\cdot\dot{\hat{p}} = 0}$$

substitute

$$= -\frac{1}{\|\vec{p}\|^2}(\dot{\vec{p}}\cdot\hat{p})\vec{p} + \frac{1}{\|\vec{p}\|}\dot{\vec{p}}$$

$$= -\frac{1}{\|\vec{p}\|}(\dot{\vec{p}}\cdot\hat{p})\hat{p} + \frac{1}{\|\vec{p}\|}\dot{\vec{p}}$$

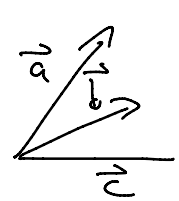
normal to $\vec{p}$ i.e., $\vec{p}\cdot\dot{\hat{p}} = 0$

$$= \frac{1}{\|\vec{p}\|}\left(\dot{\vec{p}} - (\dot{\vec{p}}\cdot\hat{p})\hat{p}\right) = \underline{\mathrm{comp}(\dot{\vec{p}}, \vec{p})}$$

Vector identities

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} \qquad \vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$$

$$\vec{a}\cdot\vec{a} = \|\vec{a}\|^2 \qquad \vec{a}\times\vec{a} = 0$$



exercise: Compute the volume of parallelpiped defined by vectors $\vec{a}, \vec{b}, \vec{c}$.