

11/29/17

Lecture #32

Rigid Body Kinetics

Euler: $\ddot{\vec{r}} = m \ddot{\vec{r}}_c = \vec{F}_{\text{net}}$

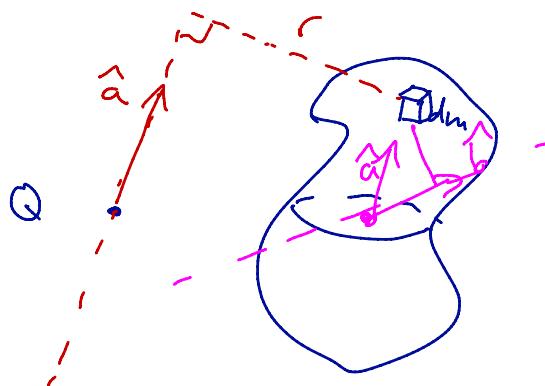
$\vec{H}_c = \vec{M}_{c,\text{net}}$

angular momentum

$\vec{H}_c = \begin{pmatrix} I_c & \vec{\omega} \end{pmatrix}$ inertia matrix



Moment of inertia: wrt point Q and an axis \hat{a}



$$I_{Q,\hat{a}} = \int_{\text{body}} r^2 d\text{dm}$$

$$\left\{ I_{c,\hat{a}} = \hat{a} \cdot (I_c \hat{a}) \right\}$$

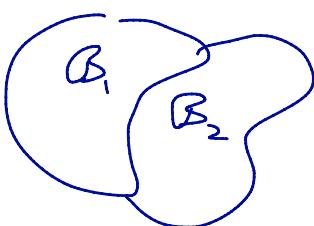
For us $\hat{a} = \hat{k}$, and Q will frequently be the center of mass.

$$I_{Q,\hat{z}\hat{z}} = I_{Q,\hat{k}\hat{k}}$$

Q=center of mass I_{zz}

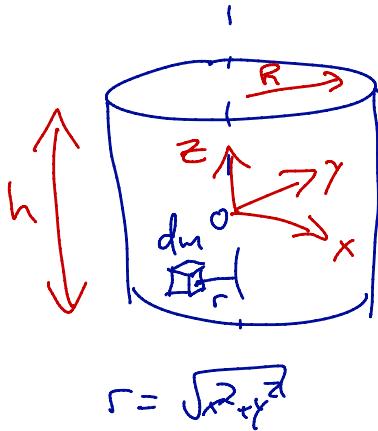
Composite bodies:

$$I_{Q,\hat{a}}^B = I_{Q,\hat{a}}^{B_1} + I_{Q,\hat{a}}^{B_2}$$



$$B = B_1 + B_2$$

example: moment of inertia of a solid cylinder $I_{c,\hat{z}}$



$$I_{c,\hat{z}} = \int_{\text{body}} (x^2 + y^2) dm = \int_0^R r^2 (h 2\pi r \rho dr)$$

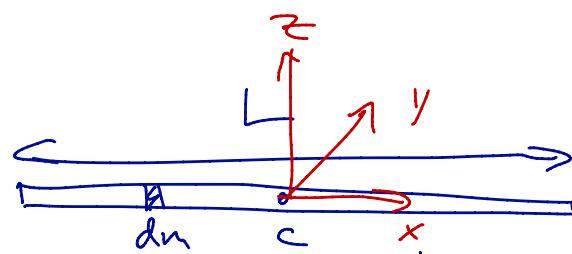
$$= \frac{\pi}{2} \rho h R^4$$

$$\rho = \frac{m}{\pi R^2 h} \Rightarrow I_{c,\hat{z}} = \frac{m}{2} R^2$$

example: thin rod

Find: $I_{c,\hat{z}}$

$$\rho = \frac{m}{L}$$

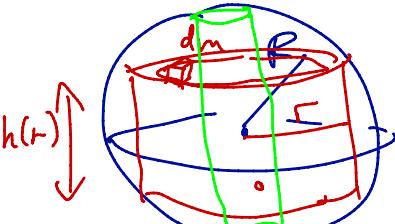


$$I_{c,\hat{z}}^{\text{rod}} = \int_{\text{rod}} x^2 dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \rho dx = \left[\frac{m}{3L} x^3 \right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

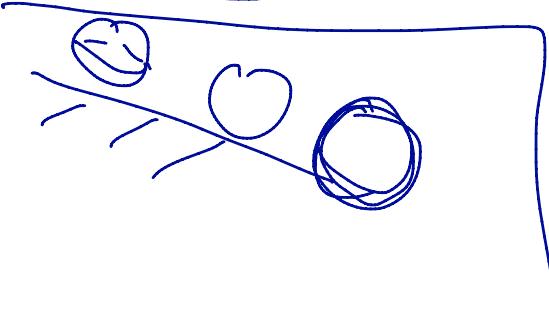
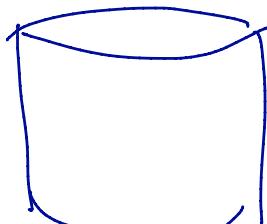
$$= \frac{m}{12} L^2$$

example:

the sphere



$$\rho = \left(\frac{m}{\frac{4}{3}\pi R^3} \right)$$

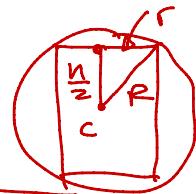


$$I_{c,\hat{z}} = \int_{\text{sphere}} r^2 dm = \int_0^R r^2 \rho (2\pi r h(r)) dr$$

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2$$

$$\Rightarrow h(r) = 2\sqrt{R^2 - r^2}$$

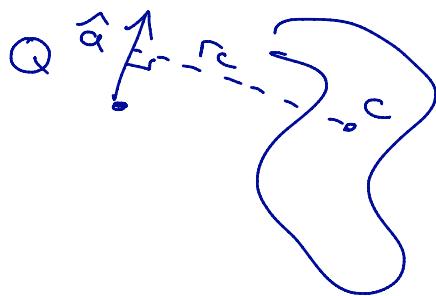
$$= 4\pi \rho \int_0^R r^3 \sqrt{R^2 - r^2} dr = \frac{2}{5} m R^2$$



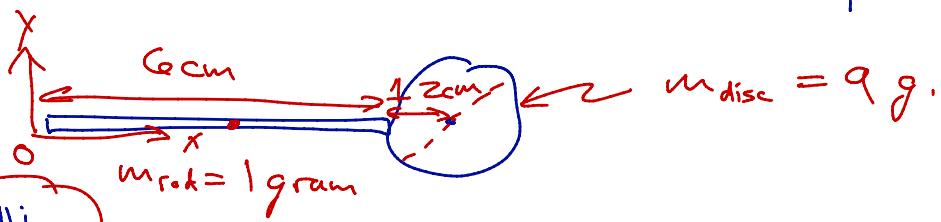
Parallel axis theorem:

$$I_{Q,\hat{a}} = m r_c^2 + I_{c,\hat{a}}$$

must be
c of m



example: moment of inertia of a flat lollipop.



find: \vec{r}_c^{lolly} , $I_{o,\hat{k}}$

solution: $\vec{r}_c^{\text{lolly}} = \left(\frac{1}{10}\right)3\hat{x} + \left(\frac{9}{10}\right)\hat{z} = \left(\frac{33}{10}\right)\hat{z}$

$$I_{o,\hat{k}} = I_{o,\hat{k}}^{\text{rod}} + I_{o,\hat{k}}^{\text{disc}}$$

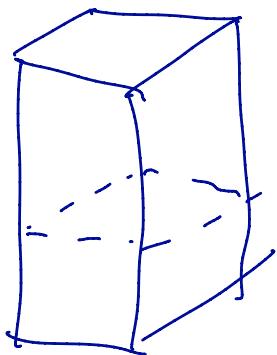
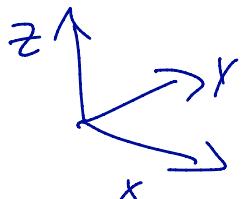
$$I_{o,\hat{k}}^{\text{disc}} = \{ \text{invoke parallel axis} \} = m_{\text{disc}} (x_c^{\text{disc}}{}^2 + y_c^{\text{disc}}{}^2)$$

$$= 9(7^2 + 0) + \frac{9}{2}(1)^2$$

$$I_{o,\hat{k}}^{\text{rod}} = m_{\text{rod}} (x_c^{\text{rod}}{}^2 + y_c^{\text{rod}}{}^2) + I_{c,\hat{k}}^{\text{rod}}$$

$$= 1(3^2 + 0^2) + \left(\frac{1}{12}\right)1 \cdot 6^2 = 12$$

Motion in the plane



Can:

- translate in the \hat{x} and \hat{z} directions
- rotate about \hat{y} -axis

Cannot:

- translate \hat{y} direction
- \hat{x} and \hat{z} axes

Coordinates:

x_c center of mass coords
 y_c
 θ axis orientation.

Euler's equation

$$\boxed{\begin{aligned} m \ddot{x}_c &= f_{x\text{net}} \\ m \ddot{y}_c &= f_{y\text{net}} \\ I_{zz} \ddot{\theta} &= M_{cz\text{net}} \end{aligned}}$$