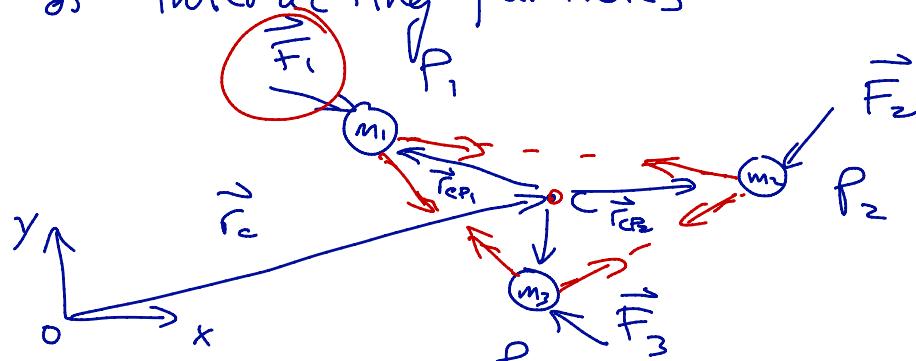


11-27-17

Lecture #31

Euler's Laws for Rigid Bodies

for a system of interacting particles



$$\vec{r}_c = \left(\frac{m_1}{m}\right)\vec{r}_{op_1} + \left(\frac{m_2}{m}\right)\vec{r}_{op_2} + \left(\frac{m_3}{m}\right)\vec{r}_{op_3}$$

$$\vec{H}_c = \underbrace{\vec{r}_{cp_1} \times (m_1 \vec{v}_{p_1})}_{H_{c1}} + \vec{r}_{cp_2} \times (m_2 \vec{v}_{p_2}) + \vec{r}_{cp_3} \times (m_3 \vec{v}_{p_3})$$

Euler's laws:

$$\begin{aligned} m\vec{r}_c &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{F}_{net} \\ \dot{\vec{H}}_c &= \vec{r}_{cp_1} \times \vec{F}_1 + \vec{r}_{cp_2} \times \vec{F}_2 + \vec{r}_{cp_3} \times \vec{F}_3 \\ &= \vec{M}_{c1} + \vec{M}_{c2} + \vec{M}_{c3} = \vec{M}_{net} \end{aligned}$$

example: consider two equal masses, connected by a massless rod, constrained to rotate about the z-axis.

Find: \vec{H}_c

$$m_1 = m_2 = m ; \vec{\omega} = \omega \hat{k}$$

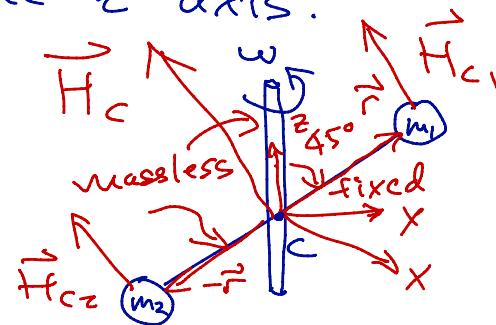
\vec{r} is the vector from the origin to m .

$$\vec{v}_1 = \vec{\omega} \times \vec{r}$$

$$\vec{H}_{c1} = \vec{r} \times (m \vec{v}_1) = m \vec{r} \times (\vec{\omega} \times \vec{r})$$

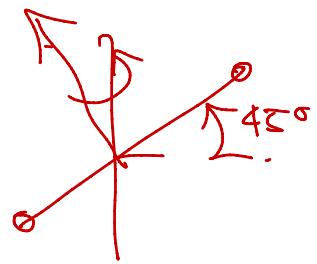
$$\vec{H}_{c2} = m (-\vec{r}) \times (\vec{\omega} \times (-\vec{r})) = \vec{H}_{c1}$$

$$\vec{H}_c = \vec{H}_{c1} + \vec{H}_{c2} = (2m) \vec{r} \times (\vec{\omega} \times \vec{r})$$



Let's put in some values:

$$\vec{\omega} = \omega \hat{k}$$



at time $t=0$:

$$\vec{r} = \hat{i} + \hat{k} \quad y = 0$$

$$\vec{H}_c = 2m((\hat{i} + \hat{k}) \times (\omega \hat{k} \times (\hat{i} + \hat{k}))) = 2m\omega(\hat{k})(\hat{i})$$

at time $t=10s$ the body has rotated 90°

$$\vec{r} = \hat{j} + \hat{k}$$

$$\vec{H}_c = 2((\hat{j} + \hat{k}) \times (\omega \hat{k} \times (\hat{j} + \hat{k}))) = 2m\omega(-\hat{j}) + (\hat{k})$$

Rigid Body

$$m \ddot{\vec{r}}_c = \vec{F}_{net}$$

$$\vec{H}_c = \vec{M}_{net}$$



What is \vec{H}_c for the rigid body.

Angular momentum for ~~the~~ an element dm .

$$\vec{r} \times (\vec{v} dm) \quad \text{element}$$

$$\begin{aligned} \vec{H}_c &= \int_{body} \vec{r} \times \vec{v} dm = \left\{ \vec{v} = \vec{v}_c + \vec{\omega} \times \vec{r} \right\} = \int_{body} \vec{r} \times (\vec{v}_c + \vec{\omega} \times \vec{r}) dm \\ &= \underbrace{\int \vec{r} \times \vec{v}_c dm}_{\vec{H}_c} + \underbrace{\int \vec{r} \times (\vec{\omega} \times \vec{r}) dm}_{\vec{I}\vec{\omega}} \end{aligned}$$

Math identity: $\vec{a} \times (\vec{b} \times \vec{a}) = (\vec{a} \cdot \vec{a} E - \vec{a} \vec{a}^T) \vec{b}$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}; \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\vec{H}_c = \left(\int_{\text{body}} \text{matrix} \right) \vec{\omega}$$

matrix of inertia

$$I_c = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

moments of inertia.

II matrix

$$\begin{bmatrix} a_2^2 + a_3^2 & -a_1 a_2 & -a_1 a_3 \\ -a_1 a_2 & a_1^2 + a_3^2 & -a_2 a_3 \\ -a_1 a_3 & -a_2 a_3 & a_1^2 + a_2^2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

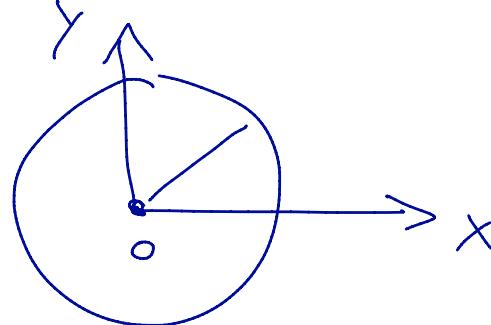
products.

$$I_{zz} = \int_{\text{body}} (x^2 + y^2) dm; I_{yy} = \int (x^2 + z^2) dm; I_{xx} = \int (y^2 + z^2) dm$$

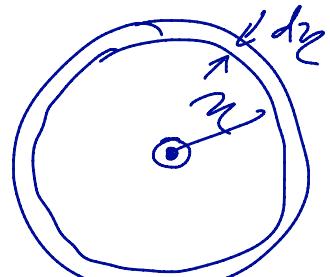
$$I_{xy} = \int_{\text{body}} -xy dm$$

In the plane we only need to concern ourselves with I_{zz} :

example:



uniform density disc
in plane

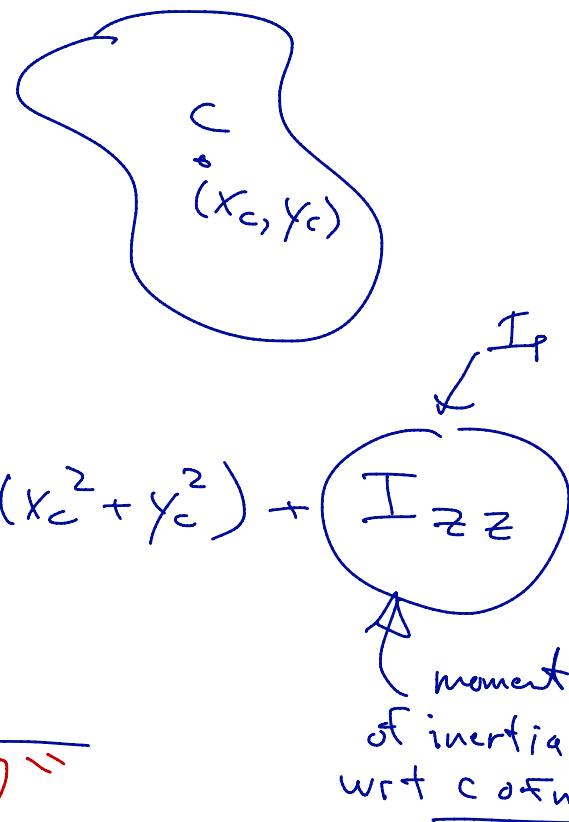
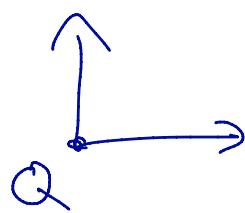


$$I_{zz} = \int_{\text{body}} (x^2 + y^2) dm = \int_0^r y^2 (2\pi y dy) \rho A \quad \text{aerial density}$$

$$= \rho 2\pi \int_0^r y^3 dy = \rho \frac{1}{2}\pi r^4 \quad \left\{ \rho = \frac{m}{\pi r^2} \right\}$$

$$I_{zz} = \frac{m}{2} r^2$$

Parallel Axis Theorem:



$$I_{Q,zz} = \int_{\text{body}} (x^2 + y^2) dm = m(x_c^2 + y_c^2) + I_{zz}$$

Composite Bodies

"glued"

moment
of inertia
wrt com

$$\beta = \beta_1 + \beta_2 + \beta_3$$



$$I_{Q,zz}^{\beta} = I_{Q,zz}^{\beta_1} + I_{Q,zz}^{\beta_2} + I_{Q,zz}^{\beta_3}$$

wrt point Q
about the z-axis