

11-16-17

Lecture #30

Euler's Second Law

single particle system

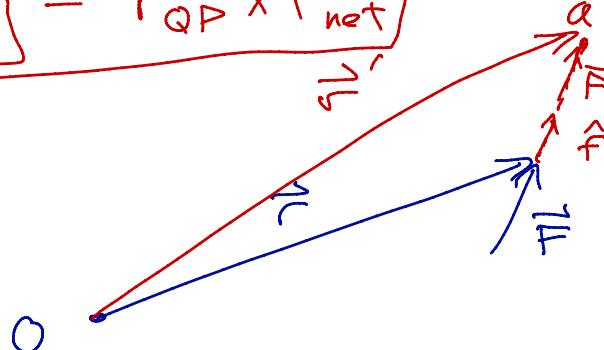
$$\vec{H}_Q^{m_i} = \vec{r}_{QP} \times (m_i \vec{v}_P)$$

angular momentum of point mass m_i wrt point Q.

Saw that if Q is stationary then

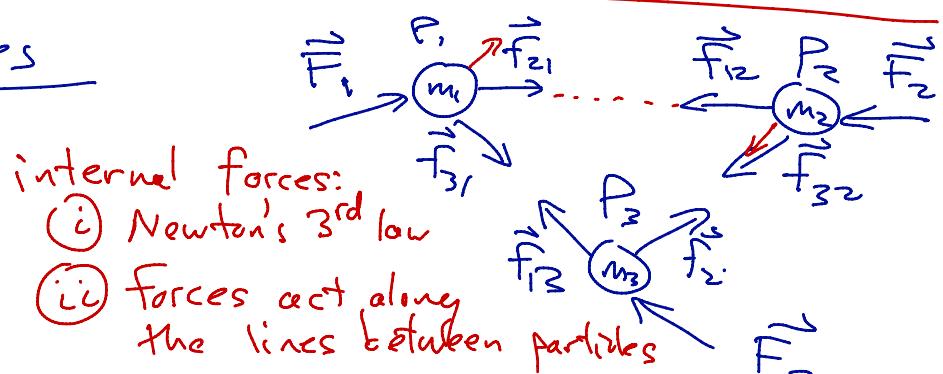
$$\boxed{\frac{d}{dt} \{ \vec{H}_Q^{m_i} \} = \vec{r}_{QP} \times \vec{F}_{\text{net}}} \quad \text{net moment about Q.}$$

Recall:



$$\begin{aligned} \vec{M}_O &= \vec{r} \times \vec{F} \\ \vec{M}_O &= \vec{r}' \times \vec{F} \\ &= (\vec{r} + \vec{a}t) \times \vec{F} \\ &= \vec{r} \times \vec{F} + \vec{a}t \times \vec{F} \end{aligned}$$

System of Particles



internal forces:
 (i) Newton's 3rd law

(ii) forces act along the lines between particles

System's angular momentum wrt point Q:

$$\begin{aligned} \text{Q-fixed } \vec{H}_Q &= \underbrace{\vec{r}_{QP_1} \times (m_1 \vec{v}_{P_1})}_{\vec{H}_Q^{m_1}} + \underbrace{\vec{r}_{QP_2} \times (m_2 \vec{v}_{P_2})}_{\vec{H}_Q^{m_2}} + \underbrace{\vec{r}_{QP_3} \times (m_3 \vec{v}_{P_3})}_{\vec{H}_Q^{m_3}} \\ \vec{H}_Q &= \vec{H}_Q^{m_1} + \vec{H}_Q^{m_2} + \vec{H}_Q^{m_3} = \text{total moments around} = \vec{r}_{QP_1} \times \vec{F}_1 + \vec{r}_{QP_2} \times \vec{F}_2 + \vec{r}_{QP_3} \times \vec{F}_3 \end{aligned}$$

When Q is a fixed point:

$$\boxed{\ddot{\vec{H}}_Q = \vec{M}_Q^{F_1} + \vec{M}_Q^{F_2} + \vec{M}_Q^{F_3}}$$

$$= \vec{M}_Q^{\text{net}}$$

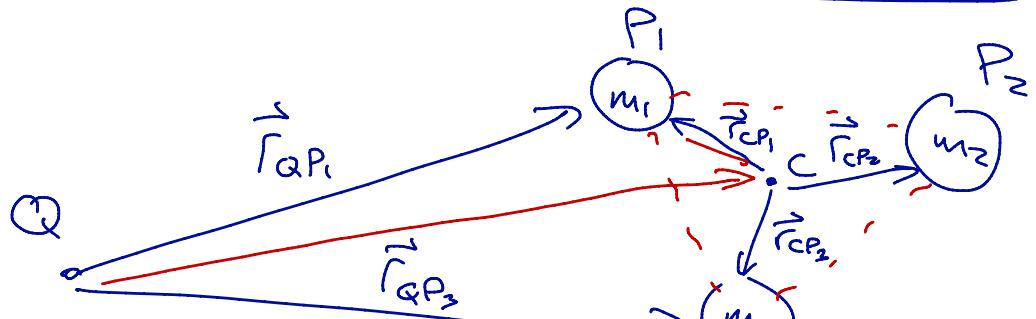
Also true when $Q = \text{center of mass}$, even if it's moving!!

Euler

$$\boxed{m\ddot{\vec{r}}_c = \dot{\vec{P}}_c = \sum \vec{F}_i \quad \begin{matrix} \uparrow \text{1st Law} \\ \ddot{\vec{H}}_c = \sum \vec{M}_{c_i} \end{matrix} \quad \begin{matrix} \uparrow \text{2nd Law} \end{matrix}}$$

Let's see why $Q = C$ works:

Angular momentum and the center of mass decomposition:



$$\begin{aligned}
 \ddot{\vec{H}}_Q &= \ddot{\vec{r}}_{QP_1} \times (m_1 \vec{v}_{P_1}) + \ddot{\vec{r}}_{QP_2} \times (m_2 \vec{v}_{P_2}) + \ddot{\vec{r}}_{QP_3} \times (m_3 \vec{v}_{P_3}) \\
 &= (\ddot{\vec{r}}_{QC} + \ddot{\vec{r}}_{CP_1}) \times (m_1 \vec{v}_{P_1}) + (\ddot{\vec{r}}_{QC} + \ddot{\vec{r}}_{CP_2}) \times \dots + \dots \\
 &= \ddot{\vec{r}}_{QC} \times (m_1 \vec{v}_{P_1} + m_2 \vec{v}_{P_2} + m_3 \vec{v}_{P_3}) + \underbrace{\ddot{\vec{r}}_{CP_1} \times m_1 \vec{v}_{P_1} + \ddot{\vec{r}}_{CP_2} \times m_2 \vec{v}_{P_2}}_{\ddot{\vec{H}}_c^m} + \ddot{\vec{r}}_{CP_3} \times m_3 \vec{v}_{P_3} \\
 &=
 \end{aligned}$$

First term

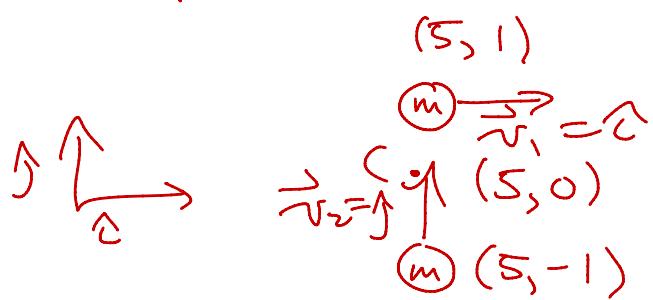
$$\ddot{\vec{r}}_{QC} \times \left(m \left(\frac{m_1}{m} \ddot{\vec{r}}_{QP_1} + \frac{m_2}{m} \ddot{\vec{r}}_{QP_2} + \frac{m_3}{m} \ddot{\vec{r}}_{QP_3} \right) \right)$$

$$= \underbrace{m \ddot{\vec{r}}_c}_{\text{m } \ddot{\vec{r}}_c}$$

$$\boxed{\ddot{\vec{H}}_Q = \ddot{\vec{r}}_{QC} \times m \ddot{\vec{r}}_c + \ddot{\vec{H}}_c}$$

$$\vec{H}_Q = \left(\begin{array}{l} \text{angular momentum} \\ \text{of the c of M wrt Q} \end{array} \right) + \left(\begin{array}{l} \text{system angular} \\ \text{momentum about c} \end{array} \right)$$

example: two particle system



Find: \vec{H}_O, \vec{H}_c

$$\vec{v}_c = \pm (\vec{v}_1 + \vec{v}_2)$$

$$\vec{H}_c = \vec{j} \times (m\vec{v}_1) + (-\vec{j}) \times (m\vec{v}_2) = -m\vec{k}$$

$$\vec{H}_O = (5\vec{c}) \times (2m^2(\vec{i} + \vec{j})) + (-m\vec{k})$$

$$= 4m\vec{k}$$

Recall that if Q is fixed

$$\vec{M}_1 + \vec{M}_2 + \vec{M}_3 = \vec{H}_Q = \frac{d}{dt} \left\{ \vec{r}_{Qc} \times m \vec{r}_{qc} + \vec{H}_c \right\}$$

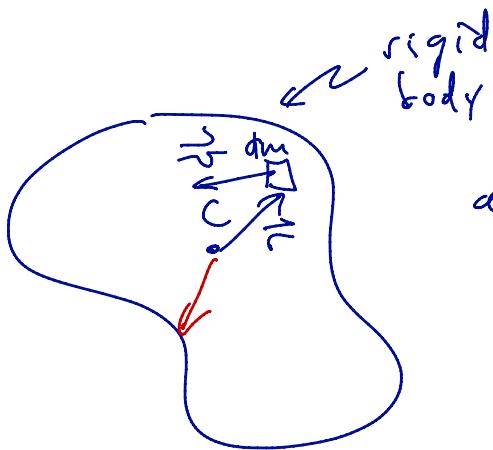
$$= \vec{r}_{Qc} \times m \vec{r}_{qc} + \vec{r}_{Qc} \times m \vec{\dot{r}}_{qc}$$

$$= \vec{r}_{qc} \times (F_1 + F_2 + F_3) + \vec{H}_c$$

$$\vec{r}_{cp_1} \times \vec{F}_1 + \vec{r}_{cp_2} \times \vec{F}_2 + \vec{r}_{cp_3} \times \vec{F}_3 = \vec{H}_c$$

$$\boxed{\sum \vec{M}_{ci} = \vec{H}_c}$$

Rigid Body equations:



$$m \ddot{\vec{r}_c} = \sum \vec{F}_i$$

$$\vec{H}_c = \sum \vec{M}_i$$

angular momentum dm wrt C
 $= \vec{r} \times (\vec{v} dm)$

$$\vec{H}_c = \int_{\text{body}} \vec{r} \times (\vec{v} dm)$$

rigid body $\vec{v} = \vec{v}_c + \vec{\omega} \times \vec{r}$

$$\Rightarrow \vec{H}_c = \int_{\text{body}} \vec{r} \times (\vec{v}_c + \vec{\omega} \times \vec{r}) dm$$

$$= \int_{\text{body}} \vec{r} \times \vec{v}_c dm + \int_{\text{body}} \vec{r} \times \vec{\omega} \times \vec{r} dm$$

$$\vec{H}_c = \int_{\text{body}} \vec{r} \times (\vec{\omega} \times \vec{r}) dm = I \vec{\omega}$$

matrix