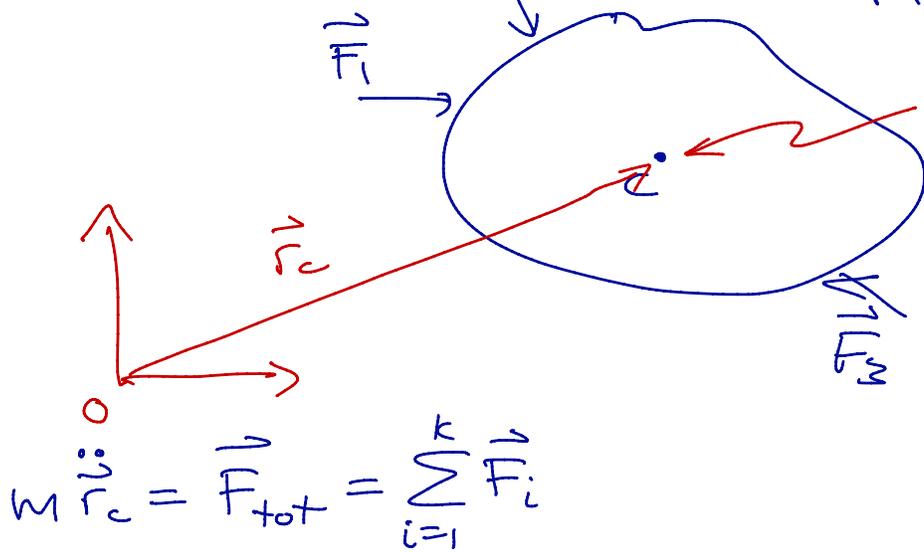


11/13/17

Lecture # 29

Recap: Euler's 1st Law



center of mass

$$\vec{r}_c = \frac{1}{m} \int_{\text{body}} \vec{r} dm$$

uniform density

$$= \frac{1}{V_{\text{tot}}} \int_{\text{body}} \vec{r} dV$$

$$m \ddot{\vec{r}}_c = \vec{F}_{\text{tot}} = \sum_{i=1}^k \vec{F}_i$$

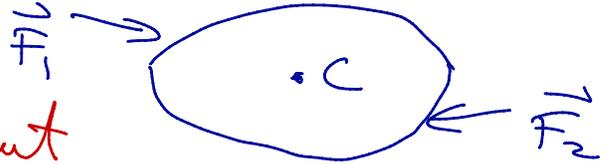
example: given a prolate spheroid ("Football")

$$m = 0.75 \text{ kg}$$

$$\vec{F}_1 = 10\hat{i} + 2\hat{j} + \hat{k} \text{ N}$$

$$\vec{F}_2 = 5\hat{i} \text{ N}$$

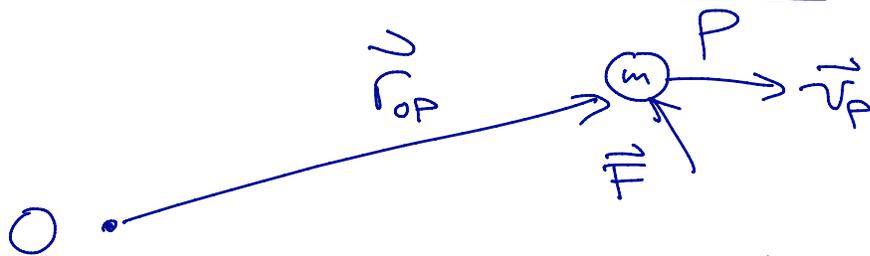
constant forces.



$$\Rightarrow \vec{r}_c(t) = \vec{r}_c(0) + \vec{v}_c(0)t + \frac{t^2}{2m}(15\hat{i} + 2\hat{j} + \hat{k})$$

What about spinning and orientation?

Recall Angular Momentum w.r.t O (fixed)



$$\vec{H}_O = \vec{r}_{OP} \times (m\vec{v}_p) \quad \text{angular momentum about } O.$$

↑ linear momentum

$$\vec{M}_O = \vec{r}_{OP} \times \vec{F} \quad \text{moment about } O.$$

$$\dot{\vec{H}}_O = \vec{M}_O$$

$$\dot{\vec{H}}_0 = \vec{M}_0$$

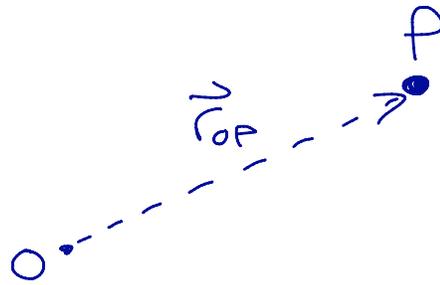
Want something like this for rigid bodies.

Aside: for single particle systems

linear momentum: $\vec{F} = 0 \implies$ momentum conserved.

angular momentum: conserved when net moments are zero.

Radial forces:

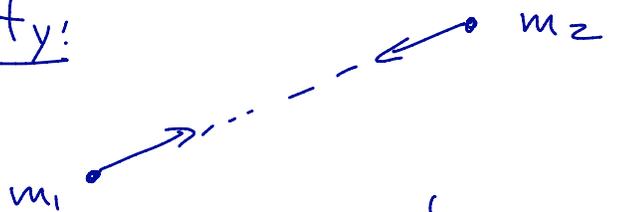


Common situation is when forces act along the direction of \vec{r}_{op} .

Spring:

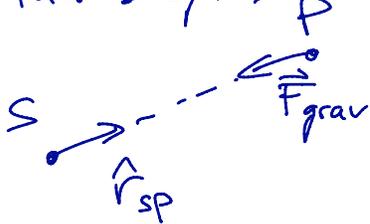


gravity:

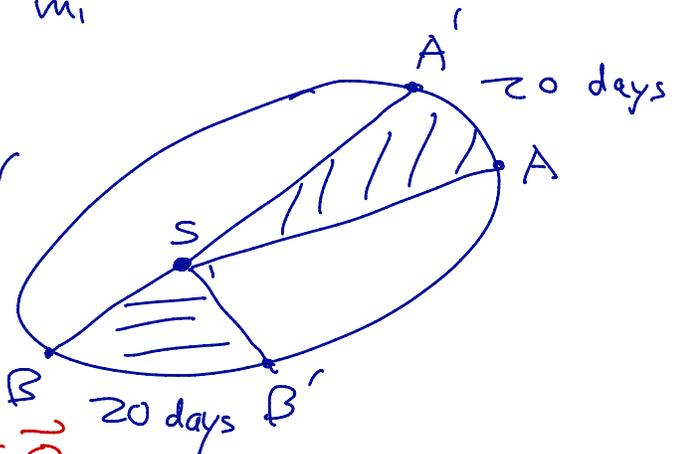


examples: Kepler's 2nd law

law says \implies area $SA A' =$ area $SB B'$

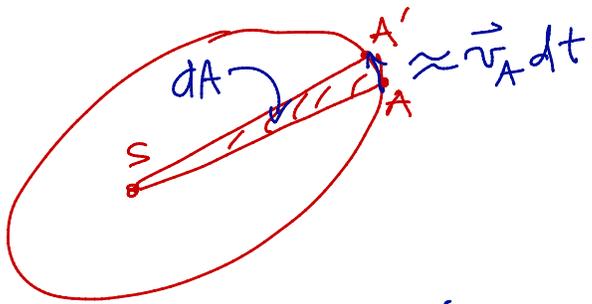


$$\vec{F}_{grav} = \frac{-GMm}{\|\vec{r}_{sp}\|^2} \hat{r}_{sp}$$



$$\vec{M}_S = \vec{r}_{sp} \times \vec{F}_{grav} = \vec{0}$$

$$\vec{H}_S = \text{const. since } \dot{\vec{H}}_S = \vec{0}.$$

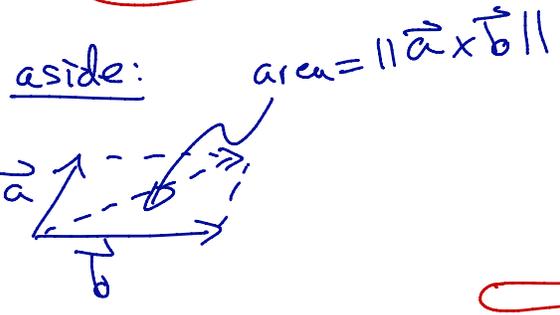


$$dA \approx \frac{1}{2} \|\vec{r}_{SA} \times (\vec{v}_A dt)\|$$

$$= \frac{dt}{2m} \|\vec{r}_{SA} \times (m\vec{v}_A)\|$$

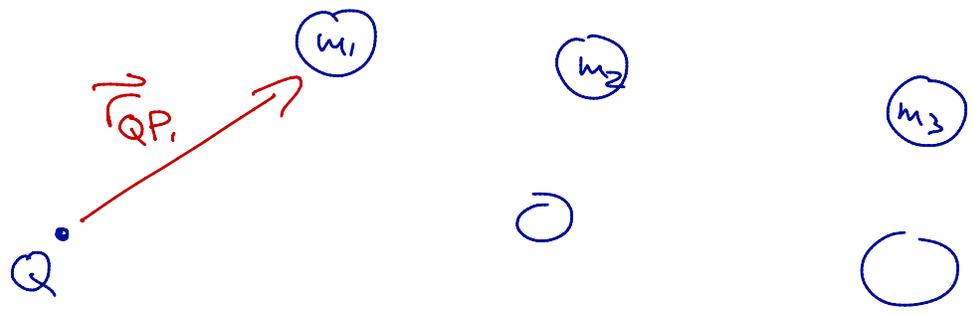
$$= \frac{dt}{2m} \|\vec{H}_S\|$$

constant.



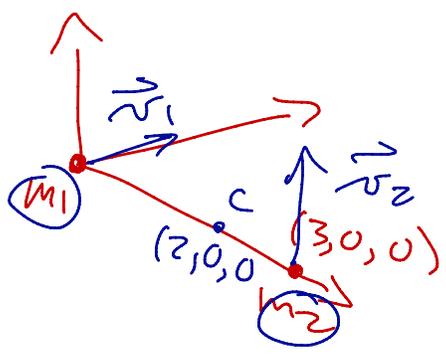
$$\Rightarrow \frac{dA}{dt} = \frac{\|\vec{H}_S\|}{2m} = \text{constant.}$$

Angular Momentum for a System of Particles



$$\vec{H}_Q = \sum_{i=1}^k \vec{r}_{QP_i} \times (m_i \vec{v}_{P_i}) = \sum_{i=1}^k \vec{H}_{Q_i}$$

example:



- $m_1 = 1 \text{ kg}$
- $\vec{r}_1 = \vec{0}$
- $\vec{v}_1 = 6\hat{j} \text{ m/s}$
- $m_2 = 2 \text{ kg}$
- $\vec{r}_2 = 3\hat{i} \text{ m}$
- $\vec{v}_2 = 3\hat{k} \text{ m/s}$

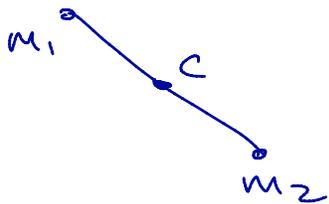
- Q1: compute \vec{H}_O
- Q2: compute \vec{H}_C , where C is the center of mass.

Question #1:

$$\begin{aligned} \vec{H}_O &= \vec{r}_1 \times (m_1 \vec{v}_1) + \vec{r}_2 \times (m_2 \vec{v}_2) \\ &= \vec{0} \times (\quad) + (3\hat{i}) \times (2 \cdot 3\hat{k}) = -18\hat{j} \end{aligned}$$

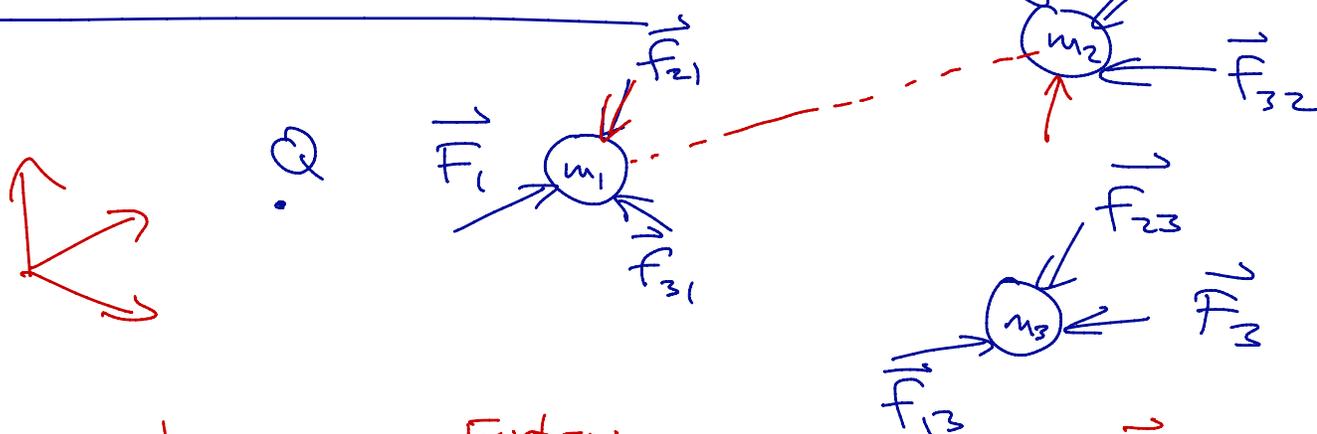
Question #2:

need \vec{r}_c : $\vec{r}_c = \frac{1}{(1+2)} (1 \cdot \vec{0} + 2(3\hat{i}))$
 $= 2\hat{i}$



$$\begin{aligned} \vec{H}_c &= \underbrace{\vec{r}_{cP_1}}_{-2\hat{i}} \times (m_1 \vec{v}_1) + \underbrace{\vec{r}_{cP_2}}_{\hat{i}} \times (m_2 \vec{v}_2) \\ &= -12\hat{k} - 6\hat{j} \end{aligned}$$

Euler's 2nd Law



Assumptions:

$$\begin{aligned} \vec{f}_{12} + \vec{f}_{21} &= \vec{0} \\ \vec{f}_{13} + \vec{f}_{31} &= \vec{0} \\ \vec{f}_{23} + \vec{f}_{32} &= \vec{0} \end{aligned}$$

Extra:

\vec{r}_{12} and \vec{f}_{12} colinear
 \vec{r}_{13} and \vec{f}_{13} colinear
 \vec{r}_{23} and \vec{f}_{23} colinear

\vec{f}_{ij} is the force i exerts on particle j

Given any point Q:

we can show with these assumptions that

total moment about point Q

$$= \vec{r}_{QP_1} \times \vec{F}_1 + \vec{r}_{QP_2} \times \vec{F}_2 + \vec{r}_{QP_3} \times \vec{F}_3$$

To see this happens let's focus on m_i :

$$\vec{F}_1, \vec{f}_{21}, \vec{f}_{31}$$

So the total moment acting around Q :

$$\vec{r}_{QP_1} \times (\vec{F}_1 + \vec{f}_{21} + \vec{f}_{31})$$

total on all particles:

$$\vec{r}_{QP_1} \times (\vec{F}_1 + \vec{f}_{21} + \vec{f}_{31}) + \vec{r}_{QP_2} \times (\vec{F}_2 + \vec{f}_{32} + \vec{f}_{12}) + \vec{r}_{QP_3} \times (\vec{F}_3 + \vec{f}_{13} + \vec{f}_{23})$$

Look at a pair:

$$\begin{aligned} & \vec{r}_{QP_1} \times \vec{f}_{21} + \vec{r}_{QP_2} \times \vec{f}_{12} \\ &= \vec{r}_{QP_1} \times \vec{f}_{21} + \vec{r}_{QP_2} \times (-\vec{f}_{21}) \\ &= (\vec{r}_{QP_1} - \vec{r}_{QP_2}) \times \vec{f}_{21} = 0 \end{aligned}$$