

#15B
Lecture #16: Examples

example: A skateboarder is traveling on a track defined $r = 2 + \cos 2\theta$. At a particular time instant: $\theta = \frac{3}{4}\pi$; $\dot{\theta} = -2 \text{ rad/sec}$, $\ddot{\theta} = -2 \text{ rad/sec}^2$

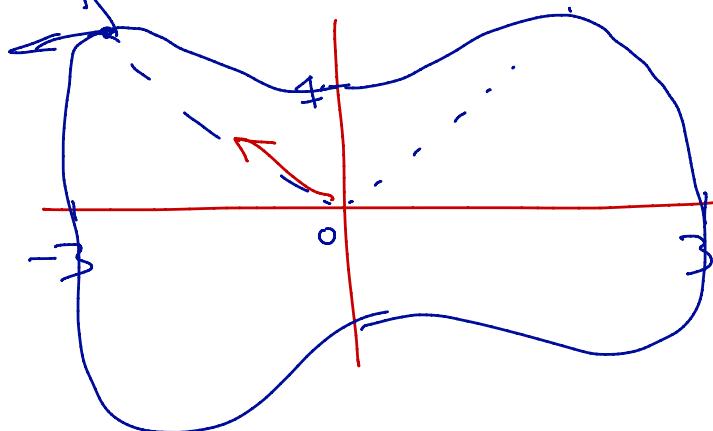
- (I) Sketch the track.
- (II) Find \vec{v} and $\vec{\alpha}$ in the polar basis.
- (III) Is she speeding up or slowing down?
- (IV) Determine the radius of curvature.

Part I:

$$\begin{array}{c} \theta \\ \hline 0 \\ \text{IV} \\ \text{III} \end{array} \quad \begin{array}{l} r \\ | \\ r=2+1=3 \\ | \\ r=2+0=2 \\ | \\ r=2-1=1 \end{array}$$

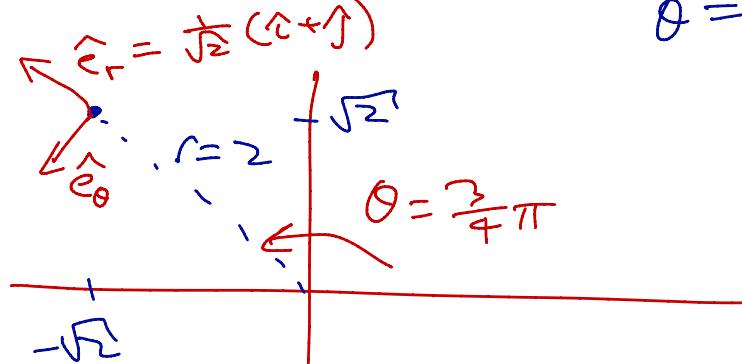
$$r = 2 + \cos 2\theta$$

Symmetry about both x and y axis.



Part II: find \vec{v} and $\vec{\alpha}$ $\theta = \frac{3}{4}\pi$, $\dot{\theta} = 2 \text{ rad/s}$, $\ddot{\theta} = 2 \text{ rad/s}^2$

Note: \hat{e}_r , \hat{e}_θ are not aligned with \hat{e}_r and \hat{e}_θ



$$\text{velocity: } \vec{r} = r\hat{e}_r \Rightarrow \vec{v} = \frac{d}{dt} \left\{ \frac{r}{\hat{e}_r} \right\} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\text{We need } \dot{r} = \frac{d}{dt} \left\{ 2 + \cos 2\theta \right\} = (-\sin 2\theta)(2\dot{\theta})$$

$$= -\sin\left(\frac{3\pi}{2}\right)(2 \cdot (-2)) = -4 \text{ m/s}$$

$$r\dot{\theta} = 2(-2) = -4 \text{ m/s} \Rightarrow \vec{v} = -4\hat{e}_r - 4\hat{e}_\theta$$

acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt} \left\{ r \hat{e}_r + r\dot{\theta} \hat{e}_\theta \right\} = (\ddot{r} - r\ddot{\theta}) \hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{e}_\theta$$

Still $\ddot{r} = \frac{d}{dt} \{ \ddot{r} \} = \frac{d}{dt} \{ -2\dot{\theta} \sin 2\theta \} = -2\ddot{\theta} \sin 2\theta - 4\dot{\theta}^2 \cos \theta$

At $\theta = \frac{3}{4}\pi$: $\ddot{r} = -2(-2) \sin(\frac{3}{2}\pi) = -4 \text{ m/s}^2$

$$\Rightarrow \vec{a} = (-4 - 2(-2)^2) \hat{e}_r + (2(-2) + 2(-4)(-2)) \hat{e}_\theta \\ = -12 \hat{e}_r + 12 \hat{e}_\theta \quad \boxed{\checkmark}$$

III Speedy up, or slowing down?

Need compute $\vec{a} \cdot \hat{e}_T$

$$\vec{a} = \ddot{s} \hat{e}_T + \frac{\dot{s}^2}{s} \hat{e}_n$$

$$\hat{e}_T = -\frac{1}{\sqrt{2}} (\hat{e}_r + \hat{e}_\theta) = \left\{ \hat{v} = \frac{1}{\|\vec{v}\|} \vec{v} \right\}$$

$$\Rightarrow \vec{a} \cdot \hat{e}_T = -12(\hat{e}_r - \hat{e}_\theta) \left(-\frac{1}{\sqrt{2}} (\hat{e}_r + \hat{e}_\theta) \right) = 0$$

Neither!

IV Obtain s :

$$\vec{a} = \frac{\dot{s}^2}{s} \hat{e}_n \quad s = \|\vec{v}\|$$

$$\Rightarrow s = \frac{\|\vec{v}\|^2}{\|\vec{a}\|} \quad \left\{ \begin{array}{l} \text{Recall that that} \\ \text{we can get } s \text{ from} \\ \vec{a} \times \vec{v} \end{array} \right\}$$

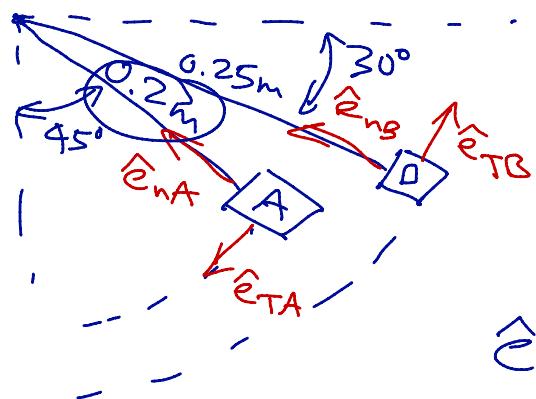
example: Suppose two cars are in a circular turn with the configuration data:

$$\dot{s}_A = 30 \text{ mph}$$

$$\dot{s}_B = 50 \text{ mph}$$

$$\ddot{s}_A = 500 \text{ mph}^2$$

$$\ddot{s} = -100 \text{ mph}^2$$



Q: find the difference in their velocities and accelerations.

$$\vec{v} = \dot{s} \hat{e}_T \quad \& \quad \vec{a} = \ddot{s} \hat{e}_T + \frac{\dot{s}^2}{r} \hat{e}_n$$

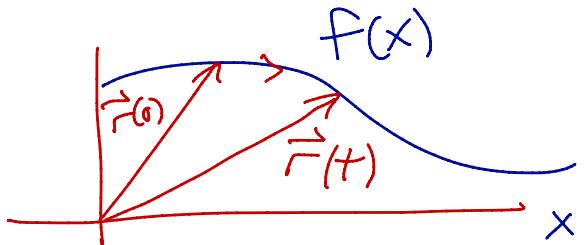
$$\hat{e}_{TA} = -\frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j}; \quad \hat{e}_{TB} = \frac{1}{2} \hat{i} + \frac{\sqrt{3}}{2} \hat{j}$$

$$\hat{e}_{nA} = -\frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j}; \quad \hat{e}_{nB} = -\frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j}$$

$$\vec{v}_A - \vec{v}_B = \dot{s}_A \hat{e}_{TA} - \dot{s}_B \hat{e}_{TB}$$

$$\vec{a}_A - \vec{a}_B = \ddot{s}_A \hat{e}_{TA} - \ddot{s}_B \hat{e}_{TB} + \frac{\dot{s}_A^2}{r_A} \hat{e}_{nA} - \frac{\dot{s}_B^2}{r_B} \hat{e}_{nB}$$

Radius of curvature along a curve



From lecture about a week ago we had the formula

$$K = \frac{1}{r} = \frac{||\vec{r} \times \vec{a}||}{||\vec{r}||^3}$$

$$\vec{r}(t) = \vec{r}_c(x(t))$$

$$\vec{r}_c(x) = x \hat{i} + f(x) \hat{j}$$

$$\vec{v}(t) = \dot{\vec{r}}(t) = \vec{r}'_c(x(t)) \dot{x}$$

$$\vec{r}'_c(x) = \hat{i} + f'(x) \hat{j}$$

$$\vec{r}''_c(x) = f''(x) \hat{j}$$

$$\vec{a}(t) = \ddot{\vec{r}}(t) = \vec{r}''_c(x(t)) \dot{x}^2 + \vec{r}'_c(x(t)) \ddot{x}$$

$$\begin{aligned}
 \vec{r} \times \vec{a} &= (\vec{r}_c' \cdot \dot{x}) \cancel{\times} (\dot{x}^2 \vec{r}_c'' + \dot{x} \vec{r}_c') \\
 &= (\vec{r}_c' \cdot \dot{x}) \cancel{\times} (\dot{x}^2 \vec{r}_c'') = \dot{x}^3 \vec{r}_c' \times \vec{r}_c'' \\
 \frac{1}{\rho} &= \frac{\|\vec{r} \times \vec{a}\|}{\|\vec{r}\|^3} = \frac{\|\dot{x}^3 \vec{r}_c' \times \vec{r}_c''\|}{\|\vec{r}_c' \dot{x}\|^3} = \cancel{\frac{\dot{x}^3 \|\vec{r}_c' \times \vec{r}_c''\|}{\dot{x}^3 \|\vec{r}_c'\|^3}} \\
 &= \frac{\|\vec{r}_c' \times \vec{r}_c''\|}{\|\vec{r}_c'\|^3} = \boxed{\frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{1}{\rho}}
 \end{aligned}$$