

09-29-17

Lecture #13

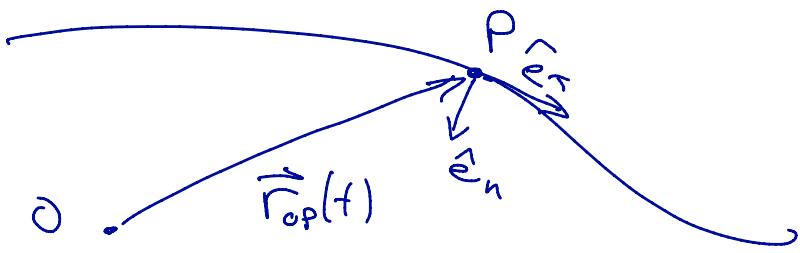
Velocity and Acceleration in Intrinsic Components

major formulas:

$$\vec{v}_p = \frac{d}{dt} \{ \vec{r}_{op}(t) \} = \dot{s}(t) \hat{e}_T(t)$$

$$\vec{a}_p = \ddot{\vec{r}}_{op} = \ddot{s}(t) \hat{e}_T + \frac{\dot{s}(t)^2}{s(t)} \hat{e}_n$$

$$\|\vec{v}\| = \dot{s}(t) \Rightarrow \hat{e}_T = \hat{v}_p = \frac{1}{\|\vec{v}\|} \vec{v}_p$$



example: at a certain instant the velocity and acceleration vectors of a particle are

$$\vec{v}_p = 4\hat{i} - 3\hat{j} \text{ m/s}$$

$$\vec{a}_p = -10\hat{i} + 20\hat{j} + 12\hat{k} \text{ m/s}^2$$

Find: (a) $\frac{d}{dt} \{ \|\vec{v}\| \} = \ddot{s}$ (b) the normal vector \hat{e}_n

$$(a) \hat{e}_T = \frac{1}{\|\vec{v}\|} \vec{v} = \frac{1}{5} (4\hat{i} - 3\hat{j})$$

$$\begin{aligned} \ddot{s} &= \vec{a} \cdot \hat{e}_T = (-10\hat{i} + 20\hat{j} + 12\hat{k}) \cdot \frac{1}{5} (4\hat{i} - 3\hat{j}) \\ &= \frac{(-40 - 60)}{5} = -20 \text{ m/s}^2 \end{aligned}$$

$$\begin{aligned} (b) \frac{\dot{s}^2}{s} \hat{e}_n &= \vec{a} - \ddot{s} \hat{e}_T = (-10\hat{i} + 20\hat{j} + 12\hat{k}) - \left(\frac{-20}{5}\right)(4\hat{i} - 3\hat{j}) \\ &= 6\hat{i} + 8\hat{j} + 12\hat{k} = \vec{g} \end{aligned}$$

$$\hat{e}_n = \hat{g} = \frac{1}{\|\vec{g}\|} \vec{g} = \frac{1}{\sqrt{6^2 + 8^2 + 12^2}} (6\hat{i} + 8\hat{j} + 12\hat{k})$$

$$\begin{aligned} \vec{v}_p &= \dot{s} \hat{e}_T \\ \vec{a}_p &= \ddot{s} \hat{e}_T + \frac{\dot{s}^2}{s} \hat{e}_n \\ &= \ddot{s} \hat{e}_T \cdot \hat{e}_T = \ddot{s} \end{aligned}$$

Question: Find a general expression for

$$\|\vec{v} \times \vec{a}\|$$

$$\vec{v} = \dot{s} \hat{e}_T$$

$$\vec{a} = \ddot{s} \hat{e}_T + \frac{\dot{s}^2}{s} \hat{e}_n$$

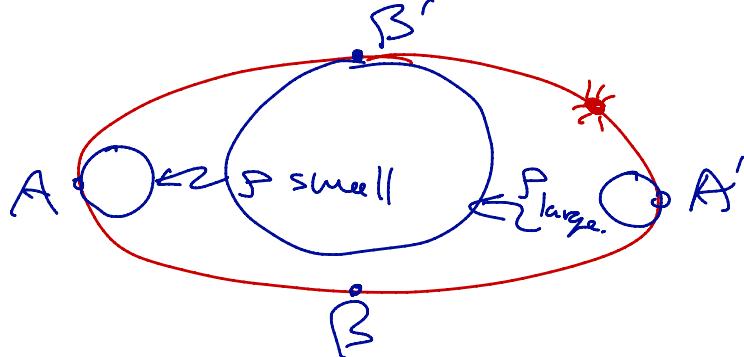
$$\vec{v} \times \vec{a} = (\dot{s} \hat{e}_T) \times \left(\ddot{s} \hat{e}_T + \frac{\dot{s}^2}{s} \hat{e}_n \right) = (\dot{s} \hat{e}_T) \times (\dot{s} \hat{e}_T) + (\dot{s} \hat{e}_T) \times \left(\frac{\dot{s}^2}{s} \hat{e}_n \right)$$

$$= \cancel{\dot{s} \ddot{s} \hat{e}_T \times \hat{e}_T} + \frac{\dot{s}^3}{s} \hat{e}_T \times \hat{e}_n$$

$$\|\vec{v} \times \vec{a}\| = \frac{1 \cdot \dot{s}^3}{s} = \frac{\|\vec{v}\|^3}{s} \stackrel{\text{in 2D}}{=} \frac{\dot{s}^3}{s} \hat{e}_T \times \hat{e}_n$$

$$\text{Equivalently } s = \frac{\|\vec{v}\|^3}{\|\vec{v} \times \vec{a}\|}$$

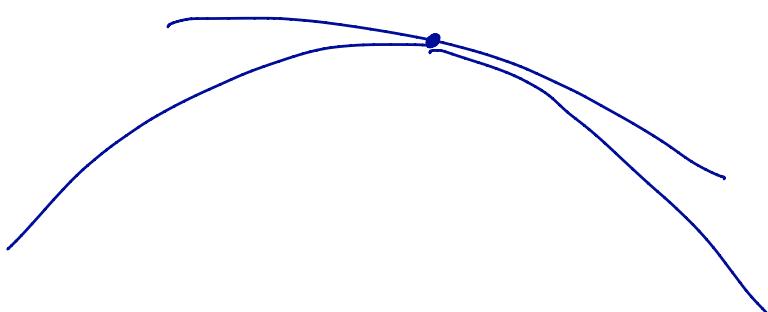
example: A bug moves along an elliptical path at a constant speed. Where along the path is the acceleration vector the largest??



$\|\vec{a}\|$ is longest at A

$\|\vec{a}\|$ shortest at B

$$\vec{a} = \frac{\dot{s}^2}{s} \hat{e}_n + \frac{\dot{s} \ddot{s}}{s} \hat{e}_T$$
$$\|\vec{a}\| = \sqrt{\frac{\dot{s}^2}{s^2} + \left(\frac{\dot{s} \ddot{s}}{s}\right)^2} = \text{const.}$$



example: an unnamed company track a car and find that $\vec{v} = 4\hat{i} + 3\hat{j}$
 $\vec{a} = 2\hat{j}$

What are ① \hat{e}_T , \hat{e}_n and s

- ② sketch the car's trajectory locally
- ③ Is the car speeding up or slowing down?

Part a: $\hat{e}_T: \vec{v} = s\hat{e}_T \Rightarrow \hat{e}_T = \hat{v} = \frac{1}{5}(4\hat{i} + 3\hat{j})$

\hat{e}_n : now $\vec{a} = \ddot{s}\hat{e}_T + \frac{\dot{s}^2}{s}\hat{e}_n$

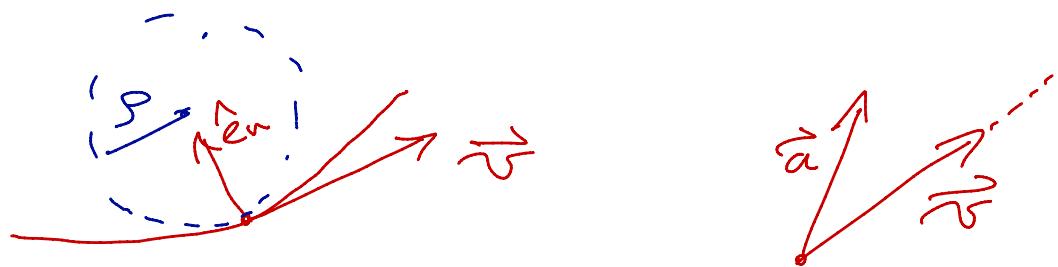
We know that $\hat{e}_T \perp \hat{e}_n$ so there are two possibilities for \hat{e}_n : either $\frac{1}{5}(3\hat{i} - 4\hat{j})$ or $\frac{1}{5}(-3\hat{i} + 4\hat{j})$

We know that $\vec{a} \cdot \hat{e}_n \geq 0$ so $\hat{e}_n = \frac{1}{5}(-3\hat{i} + 4\hat{j})$

since $(2\hat{j}) \cdot \left(\frac{1}{5}(-3\hat{i} + 4\hat{j})\right) = \frac{8}{5} = \frac{\dot{s}^2}{s}$
 $\frac{1}{5}(-6\hat{j} \cdot \hat{i} + 8\hat{j} \cdot \hat{j})$

s : $\dot{s} = \|\vec{v}\| = 5 \Rightarrow \frac{\dot{s}^2}{s} = \frac{8}{5} \Rightarrow s = \frac{125}{8}$

⑥



③ speeding up or slowing down?

Let's compute $\ddot{s} = \vec{a} \cdot \hat{e}_T \Rightarrow 2\hat{j} \cdot \left(\frac{1}{5}(4\hat{i} + 3\hat{j})\right) = \frac{6}{5} > 0$

speeding up.

$\vec{a} \cdot \vec{v}$ check sign.

example: A particle D moves along a curve with $\dot{s}(t) = 6t$ m/s, with $s(0) = 0$.

At a certain point in time t_1 , the magnitude of the acceleration vector is 12 m/s^2 and $\beta = 3 \text{ m}$. What is t_1 ?

$$\vec{a}_D = \ddot{s} \hat{e}_T + \frac{\dot{s}^2}{\beta} \hat{e}_n$$

$$\|\vec{a}_D\| = \sqrt{\ddot{s}^2 + \frac{\dot{s}^4}{\beta^2}}$$

$$\begin{aligned}s(0) &= 0 \\ \dot{s}(t) &= 6t \\ \ddot{s}(t) &= 6\end{aligned}$$

$$\beta = 3$$

$$\Rightarrow (12)^2 = 6^2 + \frac{(6t_1)^4}{3^2}, \text{ and so we get}$$

$$t_1 = 0.93 \text{ sec}$$