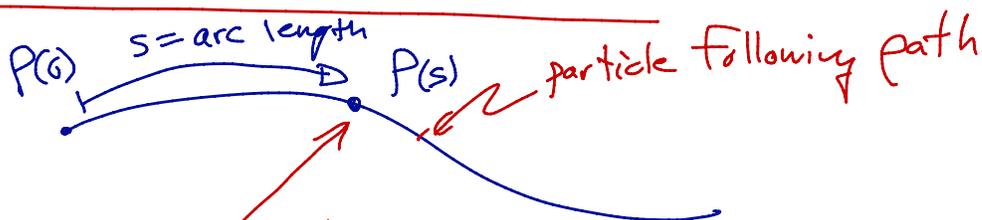


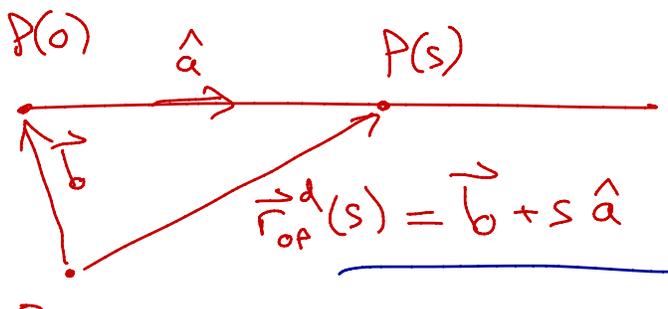
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Lecture #12

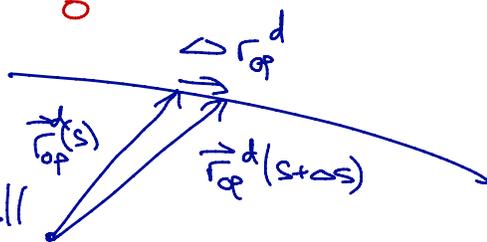
Review: the Parametric Curve



straight line example:



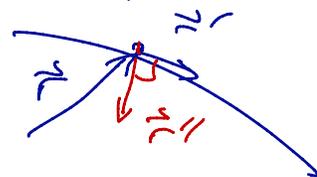
Unit tangent vectors:



$$\Delta s \approx \|\Delta \vec{r}_{op}^d\| \approx \|\vec{r}_{op}^d'(s) \Delta s\|$$

$$\Rightarrow \|\vec{r}_{op}^d'(s)\| = 1$$

Notation: $\hat{e}_T(s) = \vec{r}_{op}^d'(s)$ \leftarrow unit tangent vectors



Principal normal vector:

$$\hat{e}_T(s) \cdot \hat{e}_T(s) = 1$$

time derivative \Rightarrow

$$\hat{e}_T'(s) \cdot \hat{e}_T(s) + \hat{e}_T(s) \cdot \hat{e}_T'(s) = 0$$

$$\Rightarrow \hat{e}_T \cdot \hat{e}_T'(s) = 0$$

vectors are orthogonal.

$$\hat{e}_T'(s) = \vec{r}_{op}^d''(s) \leftarrow \text{called the curvature vector}$$

Note: for our straight line example $\vec{r}_{op}^d''(s) = 0$
no curvature.

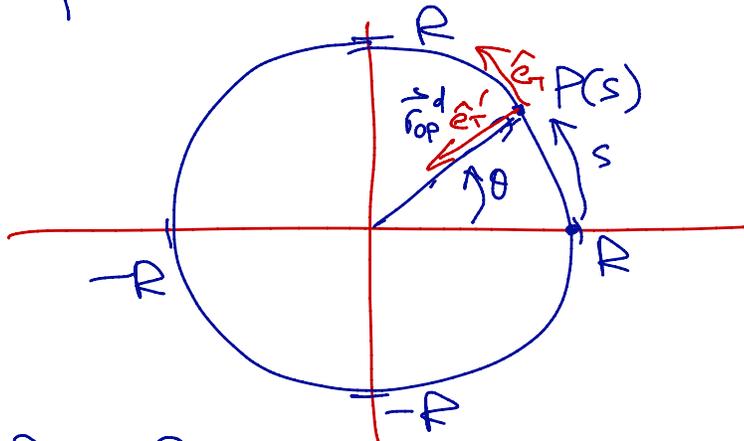
The unit normal vector as

$$\hat{e}_n(s) = \frac{1}{\|\hat{e}_T'(s)\|} \hat{e}_T'(s)$$

Curvature: $\kappa(s) = \|\hat{e}_T'(s)\|$

$$\text{Radius of curvature: } \rho(s) = \frac{1}{\kappa(s)} = \frac{1}{\|\hat{e}_T'(s)\|}$$

example: parametric curve of a circle



$$\vec{r}_{OP}^d = R \cos \theta \hat{i} + R \sin \theta \hat{j}$$

$$\vec{r}_{OP}^d(s) = R \cos\left(\frac{s}{R}\right) \hat{i} + R \sin\left(\frac{s}{R}\right) \hat{j}$$

$$\vec{r}'(s) = R \left(-\frac{1}{R} \sin\left(\frac{s}{R}\right) \hat{i} + \frac{1}{R} \cos\left(\frac{s}{R}\right) \hat{j} \right)$$

$$\hat{e}_T(s) = -\sin\left(\frac{s}{R}\right) \hat{i} + \cos\left(\frac{s}{R}\right) \hat{j} \quad \leftarrow \text{unit tangent vector}$$

$$\hat{e}_T'(s) = -\frac{1}{R} \cos\left(\frac{s}{R}\right) \hat{i} - \frac{1}{R} \sin\left(\frac{s}{R}\right) \hat{j}$$

$$= \left(-\frac{1}{R} \right) \left(\cos\left(\frac{s}{R}\right) \hat{i} + \sin\left(\frac{s}{R}\right) \hat{j} \right) \quad \leftarrow \text{curvature vector}$$

curvature: $k(s) = \|\hat{e}_T'(s)\| = \frac{1}{R} \implies \rho(s) = \frac{1}{k(s)} = R!$

$$\hat{e}_n(s) = \frac{1}{\|\hat{e}_T'(s)\|} \cdot \hat{e}_T'(s) = - \left(\cos\left(\frac{s}{R}\right) \hat{i} + \sin\left(\frac{s}{R}\right) \hat{j} \right)$$

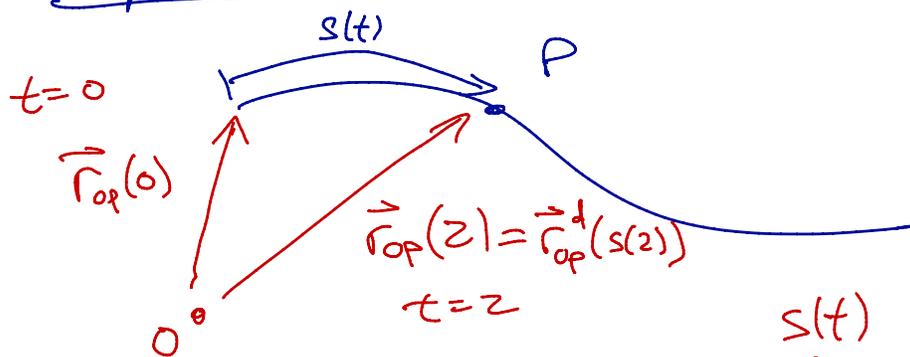
\leftarrow unit normal vector.

Note: In 2D \hat{e}_T and \hat{e}_n form a basis

3D \hat{e}_T, \hat{e}_n and $\hat{e}_n \times \hat{e}_T$ form a basis

called the natural or intrinsic basis given by the curve

Tangential and normal components with time



Set

$$\underline{\vec{r}_{op}(t) = \vec{r}_{op}^d(s(t))}$$

$s(t)$ is the distance travelled along the curve after t units of time have passed.

Velocity:

computing velocity,

$$\begin{aligned} \vec{v}_p(t) &= \frac{d}{dt} \left\{ \vec{r}_{op}(t) \right\} = \frac{d}{dt} \left\{ \vec{r}_{op}^d(s(t)) \right\} \\ &= \vec{r}_{op}^{d'}(s(t)) \dot{s}(t) = \hat{e}_T(s(t)) \dot{s}(t) \end{aligned}$$

$$\boxed{\vec{v}_p(t) = \dot{s}(t) \hat{e}_T(t)}$$

Acceleration:

Curvature vector: $\underline{\hat{e}_T^{d'}(s) = \vec{r}_{op}^{d''}(s)}$

curvature : $k(s) = \|\hat{e}_T^{d'}(s)\|$

radius : $\rho(s) = \frac{1}{k(s)}$

principal unit normal vector: $\hat{e}_n(s) = \rho(s) \vec{r}_{op}^{d'}(s)$

consider

$$\begin{aligned} \vec{a}_p &= \frac{d}{dt} \left\{ \vec{v}_p \right\} = \frac{d}{dt} \left\{ \dot{s}(t) \hat{e}_T(t) \right\} \\ &= \ddot{s}(t) \hat{e}_T(t) + \dot{s}(t) \frac{d}{dt} \left\{ \hat{e}_T(t) \right\} \end{aligned}$$

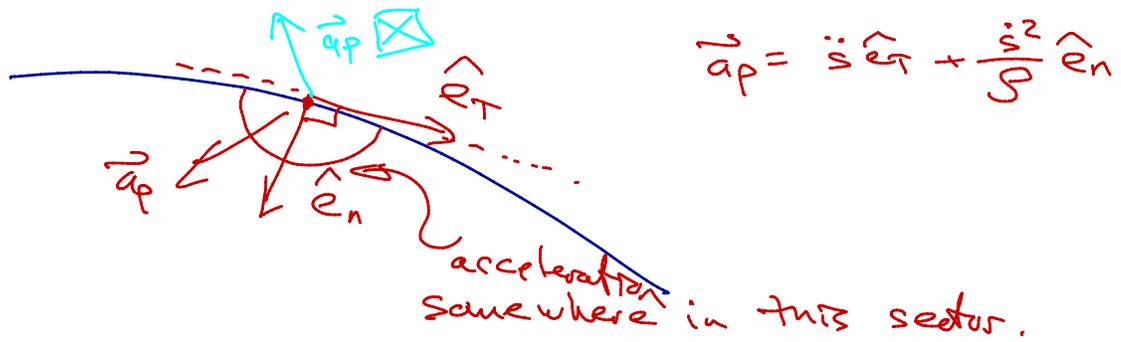
Focus on the derivative $\frac{d}{dt} \left\{ \hat{e}_T(t) \right\}$:

$$\hat{e}_T(t) = \frac{d}{dt} \left\{ \hat{e}_T^d(s(t)) \right\} = \dot{s}(t) \hat{e}_T^{d'}(s) = \dot{s}(t) \frac{1}{\rho} \cdot \hat{e}_n^d(s(t))$$

$$\Rightarrow \hat{e}_T(t) = \frac{\dot{s}}{\rho} \hat{e}_n(t)$$

$$\Rightarrow \boxed{\vec{a}_p = \ddot{s}(t) \hat{e}_T(t) + \frac{\dot{s}(t)^2}{\rho} \hat{e}_n(t)}$$

Important!!



example: A particle moves on a circle of radius 27 ft. with $s(t) = \frac{t^4}{4}$ ft.

Determine the time at which the normal and tangential components are equal in magnitude.

$$\vec{a} = \dot{s} \hat{e}_T + \frac{\dot{s}^2}{\rho} \hat{e}_n$$

Want $|\dot{s}| = \left| \frac{\dot{s}^2}{\rho} \right|$

$$\dot{s} = t^3$$

$$\dot{s} = 3t^2$$

$$\rho = 27.$$

$$\Rightarrow 3t^2 = \frac{t^6}{27} \Rightarrow \underline{t = 3}$$