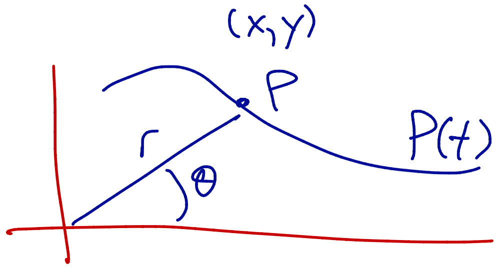
09/25/17

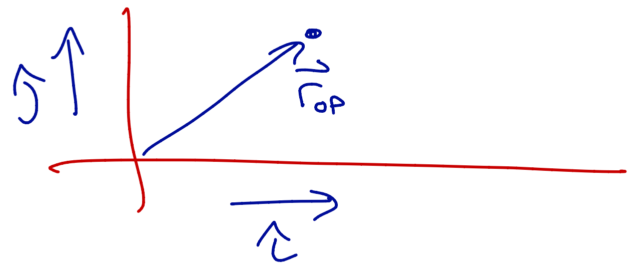
Lecture # 11

Big Picture on Position, Position vectors . . . .

positions (points in space) $\longrightarrow$ coordinates $r=2,\ \theta=60°$; $x=1,\ y=1$

position vectors $\longrightarrow$ components, wrt basis.





$$\vec{r}_{op} = \hat{\imath} + \sqrt{3}\,\hat{\jmath}$$

Velocity is the time derivative of the position vector, NOT the coordinates.

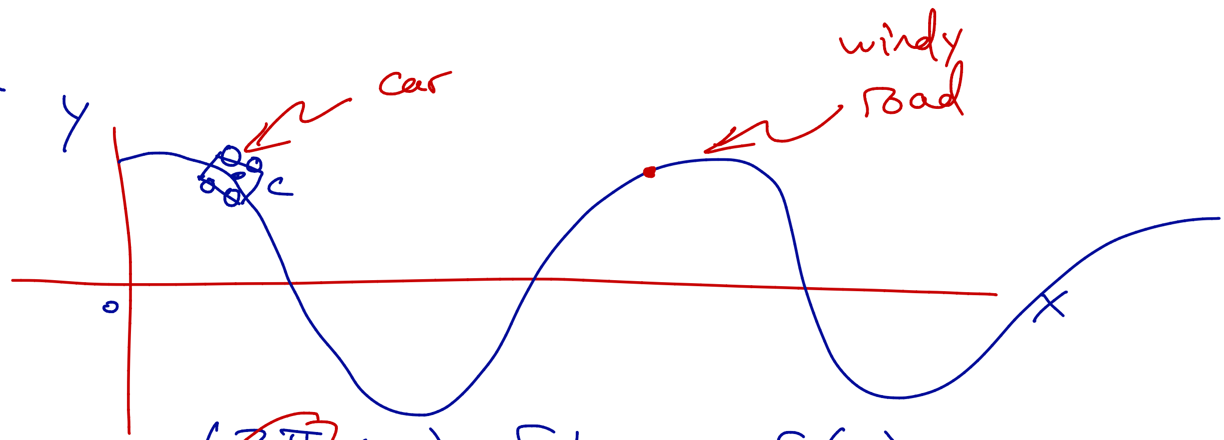
Cartesian		Polar
x, y		r, θ
$\vec{r} = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$	$\longleftrightarrow$	$\vec{r}(t) = r(t)\,\hat{e}_r(t)$
$\downarrow$ diff		$\downarrow$ diff
$\vec{v} = \dot{\vec{r}} = v_x\hat{\imath} + v_y\hat{\jmath} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}$	$\longleftrightarrow$	$\vec{v} = \dot{\vec{r}} = v_r\,\hat{e}_r(t) + v_\theta\,\hat{e}_\theta(t) = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$
$\downarrow$ diff		$\downarrow$ diff
$\vec{a} = \ddot{\vec{r}} = a_x\hat{\imath} + a_y\hat{\jmath}$	$\longleftrightarrow$	$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$

Can traverse this chart any way we like.

$$r = \sqrt{x^2+y^2} \qquad x = r\cos\theta$$
$$\theta = \tan2^{-1}(x, x) \qquad y = r\sin\theta$$

Today our focus is parametrized curves.

Example:



$$y(x) = 50\cos\left(\frac{2\pi x}{3000}\right) \text{ ft} = f(x)$$

$$f'(x) = (50)\left(\frac{2\pi}{3000}\right)\sin\left(\frac{2\pi x}{3000}\right)$$

If the driver of the car maintains a constant speed of 55 mph, what is the car's velocity at $x = 2500$ ft?

The position vector
$$\vec{r}_{oc}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} = x(t)\hat{\imath} + f(x(t))\hat{\jmath}$$

$\Rightarrow$ velocity given by
$$\vec{v}(t) = \dot{\vec{r}}_{oc}(t) = \dot{x}(t)\hat{\imath} + f'(x(t))\,\dot{x}(t)\hat{\jmath}$$

The speed being constant means $\|\vec{v}(t)\| = 55$ mph

$\Rightarrow$ Using the relationship between $\dot{x}$ & $\dot{y}$:
$$\dot{x}(t)^2 + \{f'(x(t))\cdot\dot{x}(t)\}^2 = (55\text{ mph})^2$$
$$\dot{x}(t)^2\{1+f'(x)^2\} = \dot{x}(t)^2 + \{f'(x(t))\cdot\dot{x}(t)\}^2 = \left(55\cdot\frac{88}{60}\right)^2 (\text{ft/sec})^2$$

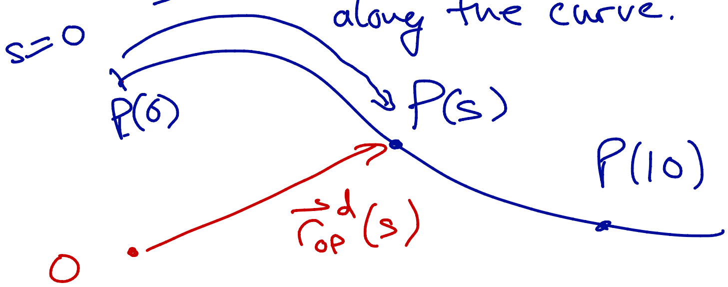
$\Rightarrow$ Let t_0 be the time at which $x = 2500$
$$\dot{x}(t_0)^2 = \frac{(80.7)^2}{1+f'(2500)^2} = \frac{(80.7)^2}{1+\left\{50\cdot\left(\frac{2\pi}{3000}\right)\sin\left(\frac{2\pi}{3000}2500\right)\right\}^2}$$

$\Rightarrow$ $\dot{x}(t_0) = 80.3$ ft/sec

$\Rightarrow$ $\vec{v}(t_0) = 80.3\hat{\imath} + 7.29\hat{\jmath}$ ft/sec

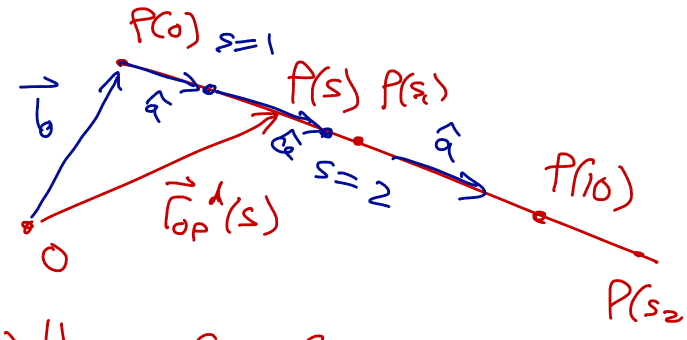
The Parametric Curve and Decompositions

s = distance travelled along the curve.



P(10) means the point we are at after travelling 10 units of distance along the curve.

example: a straight line



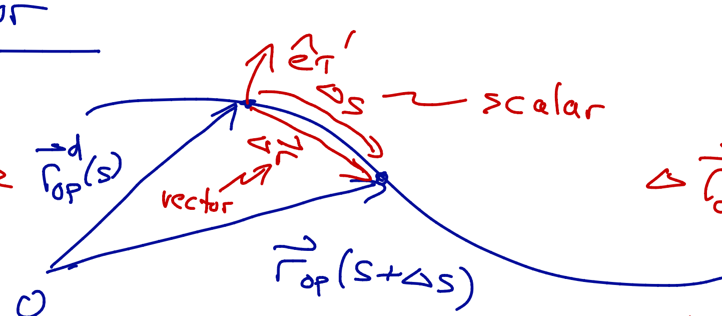
$$\vec r^{\,d}_{op}(s) = \vec b + s\cdot\hat a$$

Notice that $\|\vec r^{\,d}_{op}(s_2) - \vec r^{\,d}_{op}(s_1)\| = s_2 - s_1$

Unit tangent vector

for small Δs we have

$$(\Delta s)^2 \approx \|\Delta \vec r\|^2 \approx \|\vec r\,'(s)\Delta s\|^2$$



$$\Delta \vec r^{\,d}_{op} \approx \vec r^{\,d\,\prime}_{op}(s)\cdot\Delta s$$

$\Longrightarrow \|\vec r\,'(s)\|^2 = 1$ so $\vec r\,'(s)$ is a unit vector.

$$\boxed{\hat e_T(s) = \vec r\,'(s)}$$ $\Longleftarrow$ unit tangent vector

Curvature and the Principal Normal Vector

$$\frac{d}{ds}\left\{\hat e_T(s)\cdot\hat e_T(s)\right\} = 0 \Longrightarrow \hat e_T(s)\cdot\hat e_T'(s) = 0$$

orthogonal

$\hat e_T'(s)$ curvature vector.

$\|\hat e_T'\|$ large means lots of change locally.