TAM 212 Midterm Exam

Friday, July 15, 2016

1:00PM - 2:50PM

Please print the following information:

Full Legal Name: ________________________________

UIN: ________________________________

Instructions:

• This exam consists of 6 free response problems, worth a total of 100 points.

• The point value for each part of each problem is indicated at the beginning of that part.

• Some parts of some problems are worth bonus points. Bonus points are indicated by a plus sign (+) in front of the point value. There are a total of +10 bonus points possible.

• When appropriate, please place a box around your final answer to each problem.

• To receive full credit for a problem, you must show your work and arrive at the correct answer. No credit will be given for a correct answer with no work.

• The only items allowed (aside from your writing instrument) are one calculator and one double-sided 8.5"x11" page of personally-prepared notes.

• The following items are **NOT** allowed: books, notebooks, computers, phones.

• You may not communicate with other students during the exam.

• You must stop working at 2:50PM.

• Do not turn this page until instructed to do so.

• Failure to comply with these instructions may result in a zero grade for this exam.
The density of air at various temperatures:

<table>
<thead>
<tr>
<th>$T$ [°C]</th>
<th>$T$ [°F]</th>
<th>$\rho_{air}$ [kg/m$^3$]</th>
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Source: engineeringtoolbox.com
Problem 1

(a) (9 points) State the *general form* of Newton’s laws of motion, without any equations.

I. In an inertial reference frame, a particle acted on by no net force moves with constant velocity.

II. In an inertial reference frame, the time rate-of-change of a particle’s linear momentum is equal to the net force acting on that particle.

III. Whenever one particle exerts a force on a second particle, the second particle exerts a force on the first particle that is equal in magnitude and opposite in direction.

(b) (4 points) Does Newton’s first law follow from Newton’s second law? Explain.

*No! Law II is only valid in inertial reference frames. In order to invoke Law II, you must determine independently that your reference frame is inertial. That is where Law I comes into play: Law I serves to define/identify inertial reference frames. Having first established, using Law I, that you are in an inertial reference frame, you may then invoke Law II.*
(c) (3 points) In general, how many independent scalar equations does Newton's second law yield? How does this compare to the number of degrees of freedom of a particle?

In general, Newton's second law yields 3 independent scalar equations:

\[ \Sigma F_x = \frac{dp_x}{dt}, \quad \Sigma F_y = \frac{dp_y}{dt}, \quad \Sigma F_z = \frac{dp_z}{dt}. \]

In general, a particle has 3 degrees of freedom: one for each dimension of our (Euclidean) space.

Newton's second law gives us as many scalar equations as there are degrees of freedom for a particle, allowing us to solve for the motion of ANY particle, given all of the forces acting on it.

(d) (+5 points) While walking down the cookie aisle of your favorite grocery store, you see the following:

What is so funny about this juxtaposition?

This is funny because Isaac Newton and Gottfried Leibniz each claimed to have invented calculus independently, and their dispute was a bitter one.
Problem 2  In a vacuum, a bead of mass \( m = 6 \text{ kg} \) is moving along a guide wire whose shape is given by the function \( y(x) = 3 \sin \left( \frac{2.5x}{[m]} \right) [m] \), where \( x \) is in meters. The \( x \)-component of the particle’s velocity is constant at \( v_x = 2 \text{ m/s} \), \( x = 10 \text{ m} \). Gravity \( g = -9.8j \text{ m/s}^2 \) acts vertically. The system is shown schematically below:

(12 points) What is the force \( \vec{F}_w \) of the wire on the particle when \( x = 10 \text{ m} \)? Your answer should be a vector in the given Cartesian basis.

Given: \( m = 6 \text{ kg} \), \( y(x) = 3 \sin \left( \frac{2.5x}{[m]} \right) [m] \), \( v_x = 2 \text{ m/s} \), \( x = 10 \text{ m} \)

Find: \( \vec{F}_w = F_{w,x} \hat{i} + F_{w,y} \hat{j} \)

Solution: Invoking Newton’s second law (we assume we are working in an inertial reference frame), we have that

\[
\vec{F}_G + \vec{F}_w = m\vec{a} \Rightarrow \vec{F}_w = m\vec{a} - \vec{F}_G = m(\vec{a} + g\hat{j}) \quad (2.1)
\]

To find \( \vec{a} \), write the position vector in the given basis and differentiate twice with respect to time \( t \):

\[
\vec{r} = x \hat{i} + 3 \sin \left( \frac{2.5x}{[m]} \right) \hat{j}
\]

\[
\vec{v} = v_x \hat{i} + 7.5 v_x \cos \left( \frac{2.5x}{[m]} \right) \hat{j}
\]

\[
\vec{a} = -\frac{18.75}{[m]} v_x^2 \sin \left( \frac{2.5x}{[m]} \right) \hat{j} \quad (2.2)
\]

Substituting (2.2) into (2.1), we have that

\[
\vec{F}_w = (6 \text{ kg}) \left[ 9.8 \text{ m/s}^2 - \frac{18.75}{m} (2 \text{ m/s})^2 \sin \left( \frac{2.5 \cdot 10 \text{ m}}{m} \right) \right] \hat{j}
\]

\[
\vec{F}_w \approx (118.4 \text{ N}) \hat{j}
\]
Problem 3

(a) (5 points) State the general form of Newton’s law of universal gravitation. You may use an equation, provided you clearly define every parameter.

Consider two particles, one of mass \( m_1 \), and the other of mass \( m_2 \), separated by a distance \( r_{12} \):

\[
\begin{array}{c}
\text{\( m_1 \)} \\
\text{\( \vec{r}_{12} \)} \\
\text{\( \text{\( r_{12} \)} \)} \\
\text{\( m_2 \)}
\end{array}
\]

The force of gravity exerted on \( m_1 \) by \( m_2 \) is given by

\[
\vec{F}_G = G \frac{m_1 m_2}{r_{12}^2} \hat{r}_{12},
\]

where \( G = 6.67 \times 10^{-11} \text{ N\cdotm}^2/\text{kg}^2 \) is the universal gravitation constant and \( \hat{r}_{12} \) is the unit vector pointing from \( m_1 \) to \( m_2 \).

(b) (3 points) Explain how the above law simplifies near the surface of the earth.

Let \( M_E \) and \( R_E \) be the mass and mean radius of the earth, respectively. Near the earth’s surface, \( r_{12} \approx R_E \), and \( \hat{r}_{12} \) is approximately constant, pointing in what we would call the "downward" direction. Thus, (3.1) simplifies to

\[
\vec{F}_G \approx mg,
\]

where \( m \) is the mass of an object on the earth’s surface, and \( \vec{g} \) is the local acceleration due to gravity, which points “downward” and has a magnitude of

\[
g = G \frac{M_E}{R_E^2} \approx 9.8 \text{ m/s}^2.
\]
(c) (8 points) What is the acceleration due to gravity near the surface of the moon? The moon has a mass of $7.35 \times 10^{22}$ kg and a mean radius of 1737 km.

\[
g_{\text{Moon}} = G \frac{M_{\text{Moon}}}{R_{\text{Moon}}^2} = (6.67 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2}) \left( \frac{7.35 \times 10^{22} \text{kg}}{(1737 \times 10^3 \text{m})^2} \right)
\]

\[
g_{\text{Moon}} \approx 1.62 \text{ m/s}^2
\]

(d) (16 points) Imagine that you are an astronaut on a manned mission to the moon. You drop a baseball of mass 145 g and diameter 2.90 in from rest from a height of 20 m above the lunar surface. How long does it take the baseball to reach the ground?

Since there is no significant atmosphere on the moon, there is no drag force. The only force on the baseball is gravity. This is a case of motion with constant acceleration:

\[
y(t) = y_0 + v_{y_0}t - \frac{1}{2}g_{\text{Moon}}t^2. \quad (3.2)
\]

Let $T$ be the time at which the ball hits the ground. Then we have

\[
y_0 = 20 \text{ m}, \quad v_{y_0} = 0 \text{ m/s}, \quad y(T) = 0 \text{ m}. \quad (3.3)
\]

Substituting (3.3) into (3.2), we have

\[
0 \text{ m} = 20 \text{ m} - \frac{1}{2}g_{\text{Moon}}T^2.
\]

Solving for $T$, we find that

\[
T = \sqrt{\frac{40 \text{ m}}{g_{\text{Moon}}}} = \sqrt{\frac{40 \text{ m}}{1.62 \text{ m/s}^2}}
\]

\[
T \approx 4.96 \text{ s}
\]
Newtonian gravity is "spooky" in that it is conveyed instantaneously, allowing information to travel faster than the speed of light. Noticing this, Einstein realized that there was something wrong with Newton's theory of gravity. This led him to propose his own theory of gravity, better known as general relativity. In general relativity, gravity is not a force, but is rather the curvature of spacetime. Moving masses create ripples in the fabric of spacetime called "gravitational waves," which propagate at the speed of light, thus resolving the flaw in Newtonian gravity. Gravitational waves remained purely hypothetical concepts for nearly 100 years after Einstein predicted their existence. They were only recently observed experimentally by LIGO (Laser Interferometer Gravitational-wave Observatory), which detected the first observed gravitational waves on September 14, 2015. Einstein's theory of gravity reduces to Newton's theory in the "low-energy limit" of speeds much less than the speed of light and weak gravitational fields, such as those encountered in our solar system.
Problem 4  The tensor product \((\otimes)\) is defined as a linear operation such that:

- The tensor product of a scalar \(a\) and a tensor of arbitrary rank \(T\) is the scalar product of \(a\) and \(T\):
  \[
a \otimes T = T \otimes a = aT.
  \]  \(\text{(1)}\)

- The tensor product of any two tensors, \(S\) of rank \(m > 0\) and \(T\) of rank \(k > 0\), is a tensor of rank \(m + k\) such that, for any vector \(v\),
  \[
  (S \otimes T) \cdot v = S \otimes (T \cdot v).
  \]  \(\text{(2)}\)

(12 points) Without choosing basis vectors, derive a product rule for \(\otimes\) with respect to scalar differentiation. That is, if \(T\) and \(S\) depend on a scalar \(u\), derive a general expression for

\[
\frac{d}{du} (T \otimes S)
\]  \(\text{(3)}\)

in terms of \(T, S,\) and their derivatives with respect to \(u\). [For full credit, give a formal derivation of the product rule. For half credit, simply state the product rule.]

\[
\frac{d}{du} (\tilde{T} \otimes \tilde{S}) = \lim_{\Delta u \to 0} \frac{1}{\Delta u} \left\{ \tilde{T}(u + \Delta u) \otimes \tilde{S}(u + \Delta u) - \tilde{T}(u) \otimes \tilde{S}(u) \right\}
\]

\[
= \lim_{\Delta u \to 0} \frac{1}{\Delta u} \left\{ \tilde{T}(u + \Delta u) \otimes \tilde{S}(u + \Delta u) + \tilde{T}(u) \otimes \tilde{S}(u + \Delta u)
\right.
\]

\[
- \tilde{T}(u) \otimes \tilde{S}(u + \Delta u) - \tilde{T}(u) \otimes \tilde{S}(u) \bigg\}
\]

\[
= \lim_{\Delta u \to 0} \frac{1}{\Delta u} \left\{ \left[ \frac{\tilde{T}(u + \Delta u) - \tilde{T}(u)}{\Delta u} \right] \otimes \tilde{S}(u + \Delta u)
\right.
\]

\[
+ \tilde{T}(u) \otimes \lim_{\Delta u \to 0} \left[ \frac{\tilde{S}(u + \Delta u) - \tilde{S}(u)}{\Delta u} \right]
\bigg\}
\]

\[
= \frac{d\tilde{T}(u)}{du} \otimes \tilde{S}(u) + \tilde{T}(u) \otimes \frac{d\tilde{S}(u)}{du}
\]
Problem 5  You have seen in lecture that, for projectiles moving through a stationary fluid, the drag coefficient $C_D$ is a function only of the Reynolds number $Re$ of the fluid flow. Below is a schematic plot of $C_D$ versus $Re$ for a sphere:

![Diagram of $C_D$ vs Re]

For the group project, you have created a Microsoft Excel® spreadsheet that numerically solves the initial value problem of a spherical projectile in the presence of gravity and quadratic air resistance. This numerical solution is valid for Reynolds numbers in the range $10^3 \lesssim Re \lesssim 10^5$, for which the drag coefficient $C_D$ is more or less constant.

(a) (10 points) Using the definition of $C_D$, derive a general expression for the acceleration vector $\mathbf{a}$ of a spherical projectile near the surface of the earth in the presence of gravity and air resistance, which is valid for all Reynolds numbers. Leave $\mathbf{a}$ in terms of $C_D$.

\[
C_D = \frac{F_D}{\frac{1}{2} \rho A V^2}
\]

\[
\therefore F_D = \frac{1}{2} \rho A C_D V^2
\]

\[
\therefore \mathbf{F}_D = -\frac{1}{2} \rho A C_D V^2 \hat{V}
\]

\[
m\mathbf{a} = m\mathbf{g} + \frac{1}{2} \rho A C_D V^2 \hat{V}
\]

\[
\therefore \mathbf{a} = \mathbf{g} - \left(\frac{\rho A C_D V^2}{2m}\right) \hat{V}
\]

(b) (6 points) Explain how you could modify your spreadsheet to solve the initial value problem with this general equation of motion.

\[
a_x = -\left(\frac{\rho A C_D}{2m}\right) V_x \sqrt{V_x^2 + V_y^2}
\]

\[
a_y = -g - \left(\frac{\rho A C_D}{2m}\right) V_y \sqrt{V_x^2 + V_y^2}
\]

In the spreadsheet, add a column for $C_D$. At each time step, calculate $V$, use that to compute $Re$, and then use that to find $C_D$ from tabulated data based on the above figure. Once you know $C_D$, you can calculate the current accelerations $a_x$ and $a_y$. The rest of the numerical integration is the same.
Problem 6  Until now, we have restricted our attention to non-spinning projectiles, as spin results in an additional force: *lift*. The lift force on a spinning projectile is sometimes known as the *Magnus force* after Heinrich Gustav Magnus, who was one of the first scientists to study it experimentally. For spherical projectiles, the Magnus force may be written as

$$
\mathbf{F}_M = f_M \mathbf{\omega} \times \mathbf{v},
$$

where $f_M$ is a positive scalar, $\mathbf{\omega}$ is the angular velocity of the projectile (describing its spin rotational motion), and $\mathbf{v}$ is the velocity of the projectile's center of mass.

(a) (3 points) What is the magnitude of the Magnus force when $\mathbf{v} = \mathbf{0}$ but $\mathbf{\omega} \neq \mathbf{0}$? What does this mean, physically? Does this result make sense?

- $\mathbf{v} = \mathbf{0} \Rightarrow \| \mathbf{F}_M \| = f_M \| \mathbf{\omega} \| \| \mathbf{v} \| \sin \theta = 0$
- When a ball is rotating but not translating, the Magnus force is zero.
- This makes sense, since by symmetry there can be no preferred direction for the Magnus force in this case.

(b) (3 points) What is the magnitude of the Magnus force when $\mathbf{\omega} = \mathbf{0}$ but $\mathbf{v} \neq \mathbf{0}$? What does this mean, physically? Is this result consistent with the assumptions we have made in the lecture about projectile motion?

- $\mathbf{\omega} = \mathbf{0} \Rightarrow \| \mathbf{F}_M \| = f_M \| \mathbf{\omega} \| \| \mathbf{v} \| \sin \theta = 0$
- When a ball is translating but not rotating, the Magnus force is zero.
- This is consistent with our previous assumptions, since we assumed that we could neglect the lift force on a non-spinning projectile.

(c) (3 points) What is the magnitude of the Magnus force when $\mathbf{\omega}$ and $\mathbf{v}$ point along the same direction? What does this mean, physically? Does this result make sense?

- $\theta = 0, \pi \Rightarrow \| \mathbf{F}_M \| = f_M \| \mathbf{\omega} \| \| \mathbf{v} \| \sin \theta = 0$
- When a ball is translating along a direction that is parallel to the axis of rotation, the Magnus force is zero.
- This makes sense, since by symmetry there can be no preferred direction for the Magnus force in this case.
(d) (3 points) Shown below are snapshots of a tennis ball in flight at several instants in time, with its velocity and angular velocity indicated. It is rotating clockwise from your perspective, in the plane of the page (this is referred to as “topspin”). On each figure below, draw and label all forces of practical importance on the tennis ball. What is the effect of the Magnus force on a tennis ball with topspin?

The effect of the Magnus force on a tennis ball with topspin is to bring the ball downward faster than if the ball were not spinning.