Written Homework #2 - Solution

1. Choose a Cartesian coordinate system in which $y$ is the vertical axis and $x$ is the horizontal axis:

In the absence of air resistance, the equations of motion are as follows:

$$\ddot{x} = 0, \quad \ddot{y} = -g$$

Integrating, we find that the solution is

$$x(t) = x_0 + v_{x_0} t, \quad y(t) = y_0 + v_{y_0} t - \frac{1}{2} g t^2 \quad (1.1)$$

(a) In this case, we have the following:

$$x_0 = 0, \quad y_0 = H \quad (1.2)$$

$$v_{x_0} = V \cos \theta, \quad v_{y_0} = V \sin \theta$$

Let $T$ be the time at which you catch the ball. Then

$$y(T) = H, \quad x(T) = D \quad (1.3)$$
Substituting (1.2) and (1.3) into (1.1):

\[ D = V \cos \theta T \]  \hspace{1cm} (1.4)
\[ H = H + V \sin \theta T - \frac{1}{2} g T^2 \]  \hspace{1cm} (1.5)

Solving (1.5) for \( T \), we have

\[ T = \frac{2V \sin \theta}{g} \]  \hspace{1cm} (1.5a)

and substituting this into (1.4),

\[ D = V \cos \theta \left( \frac{2V \sin \theta}{g} \right) = \left( \frac{2V}{g} \right) V^2 \sin \theta \cos \theta \]

Hence,

\[ V^2 = \frac{gD}{2 \sin \theta \cos \theta} \]

\[ \therefore V = \sqrt{\frac{gD}{2 \sin \theta \cos \theta}} \]
(b) This time,

\[ x_0 = 0, \quad y_0 = H \]
\[ v_{x0} = U \cos \theta, \quad v_{y0} = U \sin \theta \quad (1.6) \]

Let \( T \) be the time at which you catch the ball, and suppose you run some distance \( \Delta D \) in that time. Then

\[ y(T) = W, \quad x(T) = D + \Delta D \quad (1.7) \]

Substituting (1.6) and (1.7) into (1.1),

\[ D + \Delta D = U \cos \theta T \quad (1.8) \]
\[ W = H + U \sin \theta T - \frac{1}{2} g T^2 \quad (1.9) \]

Solving (1.9) for \( T \), we have

\[ -\frac{1}{2} g T^2 + U \sin \theta T + (H-W) \]

\[ \therefore T = \frac{-U \sin \theta \pm \sqrt{U^2 \sin^2 \theta + 2g(H-W)}}{-g} \]

The quadratic formula gives us two "times" at which \( y = W \). One of them is negative, and
the other is positive. We want the positive time:

\[ T = \frac{1}{g} \left( \frac{\text{Usin} \theta + \sqrt{U^2 \sin^2 \theta + 2g(H-W)}}{1} \right) \] (1.10)

Now the average speed at which you need to run is given by

\[ S = \frac{\Delta D}{T} \]

Solving (1.8) for \( \Delta D \),

\[ \Delta D = U \cos \theta T - D \]

Thus,

\[ S = \frac{\Delta D}{T} = \frac{U \cos \theta T - D}{T} = U \cos \theta - \frac{D}{T} \] (1.11)

Substituting (1.10) into (1.11), we get

\[ S = U \cos \theta - \frac{gD}{\text{Usin} \theta + \sqrt{U^2 \sin^2 \theta + 2g(H-W)}} \]
(c) From part (a),

\[ V = \frac{(32.2 \text{ ft/s}^2)(60 \text{ ft})}{\sqrt{2 \sin 50^\circ \cos 50^\circ}} \approx 44.2923 \text{ ft/s} \]

Converting this into mph,

\[ V = (44.2923 \text{ ft/s}) \left( \frac{1 \text{ mi}}{5280 \text{ ft}} \right) \left( \frac{3600 \text{ s}}{1 \text{ hr}} \right) \approx 30.2 \text{ mph} \]

From part (b), after substituting and converting, we find that

\[ S \approx 6.24 \text{ mph} \]
2. Similarly to Problem 1, choose $x$ and $y$ such that

![Graph showing motion](image)

(a) Without air resistance, (1.1) still holds.

(i) Just as before, we get (1.5a):

$$T = \left( \frac{2 \left( \frac{50 \text{ ft/s}}{32.2 \text{ ft/s}^2} \right) \sin 40^\circ}{5.280 \text{ ft}} \right) \left( \frac{1 \text{ hr}}{3600 \text{ s}} \right)$$

$$\therefore T \approx 2.93 \text{ s} \quad \square$$

(ii) From (1.4),

$$D = \left( 50 \frac{\text{ ft}}{\text{ s}} \right) \cos 40^\circ \left( 2.93 \text{ s} \right) \left( \frac{5.280 \text{ ft}}{\text{ hr}} \right) \left( \frac{1 \text{ hr}}{3600 \text{ s}} \right)$$

$$\therefore D \approx 164 \text{ ft} \quad \square$$
(iii) The maximum height occurs at \( t = \frac{T}{2} \).

Using (1.1),

\[
y_{\text{max}} = (50 \text{ mph}) \sin 40^\circ \left( \frac{\frac{2.93 \text{s}}{2}}{2} \right) \left( \frac{5.280 \text{ ft}}{3600 \text{s}} \right) - \frac{1}{2} \left( 32 \text{ ft/s}^2 \right) \left( \frac{2.93 \text{s}}{2} \right)^2
\]

\[
: y_{\text{max}} \approx 34.5 \text{ ft} \quad \Box
\]

Alternatively, set \( v_y = v_{yo} - gt \) equal to zero, then solve for the time at which \( y \) is maximum and plug that into (1.1) to get the same answer.

(iv) From (1.1),

\[ v_x = v_{xo} = \text{constant} \]
\[ v_y = v_{yo} - gt \]

At time \( t = T = \frac{2v \sin \theta}{g} = \frac{2v_{yo}}{g} \), we find that

\[ v_x(T) = v_{xo} \]
\[ v_y(T) = v_{yo} - gT = v_{yo} - 2v_{yo} = -v_{yo} \]

\[
: v(T) = \sqrt{v_{xo}^2 + (-v_{yo})^2} = v_0 = 50 \text{ mph} \quad \Box
\]

No need to invoke the conservation of energy!
(b) The only way to find the answers is to solve the equations of motion with quadratic air resistance numerically, like in Worksheet #2. In what follows, I am using the Excel spreadsheet I made with the following parameters:

\[ m = 0.430 \text{ kg} \]
\[ \rho_{\text{air}} = 1.204 \text{ kg/m}^3 \text{ @ 68}^\circ \text{F} \]
\[ D = 0.22 \text{ m} \]
\[ c = \frac{\pi}{16} \rho_{\text{air}} D^2 = \frac{\pi}{16} (1.204 \text{ kg/m}^3)(0.22 \text{ m})^2 \approx 0.0114 \text{ kg/m} \]
\[ g = 9.8 \text{ m/s}^2 \]
\[ x_0 = y_0 = 0 \text{ m} \]
\[ v_{x_0} = (50 \text{ mph}) \cos 40^\circ \approx 17.1 \text{ m/s} \]
\[ v_{y_0} = (50 \text{ mph}) \sin 40^\circ \approx 14.4 \text{ m/s} \]
\[ \Delta t = 0.001 \text{ s} \]

(i) I find that, when \( y \) becomes zero for the second time, \( t \approx 2.421 \text{ s} \). This is about half a second less than it was in part (a). □

(ii) Again, when \( y \) becomes zero for the second time, \( x \approx 26.84 \text{ m} \approx 88 \text{ ft} \). This is about half as far as in part (a) ! □
(iii) When \( v_y = 0 \), I find that
\[
y \approx 7.26 \text{ m} \approx 23.8 \text{ ft}, \text{ roughly } 2/3 \text{ of the answer from part (a)}. \quad \Box
\]

Note that in this case you cannot use the value of \( y \) at time \( t = \frac{1}{2} (2.421 \text{ s}) \), because the trajectory is no longer symmetric.

(iv) When \( y \) becomes zero for the second time, I find that \( v_x \approx 7.41 \text{ m/s} \) and \( v_y \approx -10.47 \text{ m/s} \), so the total speed comes out to
\[
v \approx 12.8 \text{ m/s} \approx 28.6 \text{ mph}, \text{ nearly half the original speed of 50 mph}. \text{ Note that in this case, mechanical energy is not conserved. Where does this lost energy go?} \quad \Box
\]

Below is a sketch comparing parts (a) and (b).
(c) The earth has a radius of

\[ R_E = \left( 6.37 \times 10^6 \text{ pr} \right) \left( \frac{100 \text{ cm}}{\text{ pr}} \right) \left( \frac{1 \text{ in}}{2.54 \text{ cm}} \right) \left( \frac{1 \text{ ft}}{12 \text{ in}} \right) \approx 2.09 \times 10^7 \text{ ft} \]

During the time the soccer ball was in the air, the earth rotated by an amount \( \theta \) given by

\[ \theta = \Omega_{\text{earth}} T, \quad (2.1) \]

where \( T \approx 2.42 \text{ s} \) and

\[ \Omega_{\text{earth}} = \left( \frac{2\pi \text{ rad}}{24 \text{ hr}} \right) \left( \frac{1 \text{ hr}}{3600 \text{ s}} \right) \approx 7.3 \times 10^{-5} \text{ rad/s} \]

The arc length swept out in that time is given by

\[ s = R_E \theta = R_E \Omega_{\text{earth}} T \]
\[ = (2.09 \times 10^7 \text{ ft}) (7.3 \times 10^{-5} \text{ rad/s}) (2.42 \text{ s}) \]

\[ \therefore s \approx 3,680 \text{ ft} \]

This is roughly \( 2/3 \) of a mile!
(d) While this distance seems large to us, consider the angle $\Theta$ swept out, as given by (2.1), while the ball is in the air:

$$
\Theta = \Omega_{\text{earth}} T = (7.3 \times 10^{-5} \text{ rad/s})(2.4215) \approx 0.000176 \text{ rad}
$$

This is about 0.01°, so not very big at all.

Because this angle is so small, while the ball is in the air, our reference frame is essentially moving in a straight line at a constant speed of $R \Omega_{\text{earth}}$, and it barely rotates at all.

Recall that an inertial reference frame is one which neither accelerates nor rotates. Thus, for times on the order of the flight of a soccer ball, the earth is, to a very good approximation, an inertial reference frame.

Note that, over longer periods of time, this approximation breaks down, and we have to take into account centrifugal and Coriolis effects.
3. (a) In the absence of air resistance, we have the following equations of motion:

\[ \ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} = -g \]

The general solution is

\[
\begin{align*}
x(t) &= x_0 + v_{x0} t \\
y(t) &= y_0 + v_{y0} t \\
z(t) &= z_0 + v_{z0} t - \frac{1}{2} gt^2
\end{align*}
\]

Let \( T \) be the time at which the ball is directly over the net. Then we have the following:

\[
\begin{align*}
x_0 &= 19 \text{ ft}, \quad y_0 = 0 \text{ ft}, \quad z_0 = 7 \text{ ft} \\
v_{x0} &= -20.5 \text{ ft/s}, \quad v_{y0} = 87.5 \text{ ft/s}, \quad v_{z0} = 0 \text{ ft/s}
\end{align*}
\]

and that \( y(T) = 39 \text{ ft} \). Thus,

\[
39 \text{ ft} = (87.5 \text{ ft/s}) T
\]

\[
\therefore \quad T = \frac{39 \text{ ft}}{87.5 \text{ ft/s}} \approx 0.4465
\]

\[
z(T) = 7 \text{ ft} - \frac{1}{2} (32.2 \text{ ft/s}^2)(0.4465)^2 \approx 3.8 \text{ ft}
\]
(b) Note that the motion of the tennis ball is confined to a plane. Denote by "d" the horizontal coordinate in this plane (the vertical coordinate is still z):

The value of d at which the ball is directly over the net can be found using geometry:

\[ d_{net} \sin \theta = 39 \text{ ft} \rightarrow d_{net} = \frac{39 \text{ ft}}{\sin \theta} \]
The angle $\theta$ can be determined from the given initial velocity components:

\[ \vec{V}_0 = (-20.5 \text{ ft/s})\hat{i} + (87.5 \text{ ft/s})\hat{j} \]

\[ \tan\theta = \frac{87.5 \text{ ft/s}}{-20.5 \text{ ft/s}} \rightarrow \theta = \tan^{-1}\left(\frac{87.5}{20.5}\right) \approx 1.34 \text{ rad} \]

\[ \therefore d_{\text{net}} = \frac{39 \text{ ft}}{\sin(1.34 \text{ rad})} \approx 40.06 \text{ ft} \approx 12.21 \text{ m} \]

I use my spreadsheet with the following parameters:

\[ m = 0.058 \text{ kg} \]

\[ \rho_{\text{air}} = 1.204 \text{ kg/m}^3 @ 20^\circ \text{C} \]

\[ D = 2.70 \text{ in} \approx 0.0686 \text{ m} \]

\[ c = \frac{\pi}{10} \rho_{\text{air}} D^2 \approx 0.00112 \text{ kg/m} \]

\[ g = 9.8 \text{ m/s}^2 \]

\[ d_0 = 0 \text{ m} \]

\[ z_0 = 7 \text{ ft} \approx 2.1336 \text{ m} \]

\[ V_{d0} = \sqrt{(-20.5 \text{ ft/s})^2 + (87.5 \text{ ft/s})^2} \approx 89.9 \text{ ft/s} \approx 27.4 \text{ m/s} \]

\[ V_{z0} = 0 \text{ m/s} \]

\[ \Delta t = 0.001 \text{ s} \]
I find that, when \( d = d_{\text{net}} \approx 12.21 \text{ m} \),
\[ z \approx 0.99 \text{ m} \approx 3.25 \text{ ft}, \]
This is a little more than \( 1/2 \) a foot lower than in part (a),
but it is still greater than 3 ft. In fact, the tennis ball is about 3 inches above the net, and it has a radius of 1.35 in, so it clears the net without touching it. \( \square \)

Note that the net is slightly higher than 3 ft at the two ends, but even so, the ball will likely clear the net (assuming, of course, that there is no "top spin" on the ball, which would cause it to be lower).

\[ \begin{array}{c}
\omega \\
C \\
m g \\
V
\end{array} \]

Top Spin: \( \omega \) positive as shown
Back Spin: \( \omega \) negative as shown
4. (a) \( M_{\text{Mars}} = 6.39 \times 10^{23} \text{ kg} \)
\( R_{\text{Mars}} = 3.397 \times 10^6 \text{ m} \)
\[ \text{atmosphere: } 95\% \text{ CO}_2 \]

(b) \hspace{1cm} \text{Newton's Law of Gravity:}
\[ F_g \propto G \frac{M_{\text{Mars}} m}{R_{\text{Mars}}^2} \]
\[ \Rightarrow g_{\text{Mars}} = \frac{G M_{\text{Mars}}}{R_{\text{Mars}}^2} \]
\[ g_{\text{Mars}} = (6.67 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}) (6.39 \times 10^{23} \text{ kg}) (3.397 \times 10^6 \text{ m})^{-2} \]
\[ \therefore g_{\text{Mars}} \approx 3.69 \text{ m/s}^2 \]

(c) \text{Given:}
\[ m = 0.145 \text{ kg} \]
\[ D = 2.90 \text{ in} \approx 0.0737 \text{ m} \]
\[ \rho_{\text{CO}_2} \approx 0.020 \text{ kg/m}^3 \] \("\text{Mars Fact Sheet, NASA}\"
\[ c = \frac{\pi}{4} \rho_{\text{CO}_2} D^2 \approx 0.000021 \text{ Kg/m} \]
\[ g_{\text{Mars}} \approx 3.69 \text{ m/s}^2 \]
\[ y_0 = 20 \text{ m} \]

\text{Find: } T > 0 \text{ such that } y(T) = 0
(purely vertical)

By integrating the equation of motion for a projectile in the presence of gravity and quadratic air resistance (as you are asked to do in Part 1 (c) of the group project), you should have found that

\[ y(T) = y_0 - \frac{m}{c} \ln \left[ \cosh \left( \sqrt{\frac{cg}{m}} T \right) \right] \]

Setting \( y(T) = 0 \) and solving for \( T \), we find that

\[ T = \sqrt{\frac{m}{cg}} \cosh^{-1} \left( e^{\frac{cy_0}{m}} \right) \]

Substituting the given values, we obtain

\[ T = \sqrt{\frac{0.145 \text{ kg}}{(2.1 \times 10^{-5} \text{ kg/m})(3.69 \times 1/\text{s}^2)}} \cosh^{-1} \left( e^{\frac{(2.1 \times 10^{-5} \text{ kg/m})(20 \text{ m})}{0.145 \text{ kg}}} \right) \]

\[ \therefore T \approx 3.2925 \text{ s} \]

Note that, if we had neglected air resistance, we would have gotten \( \approx 3.2909 \text{ s} \), which is almost the same.