TAM 212 Worksheet: Numerical Integration in Excel®
(with an Application to Basketball)

Solutions

In this worksheet, you will create a Microsoft Excel® spreadsheet that solves the initial value problem of a projectile near the surface of the earth in the presence of quadratic air resistance. You will then use your spreadsheet to “make” a basket from the three-point line of a regulation NBA basketball court.

Analysis

Recall that the acceleration of an object in the presence of gravity and quadratic air resistance is given by

\[ \mathbf{a} = \mathbf{g} - \frac{c}{m} \mathbf{v} \mathbf{v}, \]  

where \( \mathbf{a} \) is the object’s acceleration vector, \( \mathbf{g} \) is the acceleration due to gravity near the surface of the earth, \( c \) is the so-called drag parameter, \( m \) is the object’s mass, \( \mathbf{v} \) is the object’s speed, and \( \mathbf{v} \) is the object’s velocity vector.

1. Choosing a Cartesian coordinate system in which the \( y \)-axis points vertically upward (away from the ground), decompose (1) into its \( x \)- and \( y \)-components. That is, express both \( a_x \) and \( a_y \) in terms of \( g \) (the magnitude of \( g \)), \( c \), \( m \), \( v_x \), and \( v_y \).

\[ a_x = -\frac{c}{m} v_x \sqrt{v_x^2 + v_y^2} \]
\[ a_y = -g - \frac{c}{m} v_y \sqrt{v_x^2 + v_y^2} \]

2. The above equations constitute the equations of motion for the projectile. In order to solve these equations for the object’s position \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \) as a function of time \( t \), we must specify initial conditions for certain quantities. Which quantities must we specify initial conditions for?

The initial position: \( x(0) \) and \( y(0) \), and the initial velocity: \( v_x(0) \) and \( v_y(0) \)

Numerical Integration

Taken together, the equations of motion and initial conditions constitute an initial value problem. In rare cases, initial value problems can be solved analytically, and closed-form expressions for all of the unknowns can be found. Most of the time, however, no such analytical solution exists, and the initial value problem must be solved numerically, with specific numerical values for the initial conditions. There are many numerical integration schemes that have been developed to solve initial value problems. Here, we will use the explicit Euler method. The explicit Euler method starts by discretizing the independent variables.
3. What are the independent variables in our case?

There is only one independent variable: time $t$

At each time step $t_n$, we know the current position of the object $(x_n, y_n)$ and the current velocity of the object $(v_{x,n}, v_{y,n})$. Using this information, we can compute the current acceleration of the object $(a_{x,n}, a_{y,n})$ using the equations of motion.

What we want is to find the position $(x_{n+1}, y_{n+1})$ and velocity $(v_{x,n+1}, v_{y,n+1})$ of the object at the next time step: $t_{n+1} = t_n + \Delta t$, where $\Delta t$ is a constant time increment that we choose. This is done by approximating all derivatives with finite differences.

For example, acceleration is defined as

$$a = \frac{dv}{dt}.$$

We may therefore write the following approximation:

$$a_n \approx \frac{\Delta v}{\Delta t} = \frac{v_{n+1} - v_n}{\Delta t}.$$ 

Note that, the smaller $\Delta t$ is, the more accurate this approximation is, and therefore the more accurate our numerical solution will be. Rearranging this equation, we find that

$$v_{n+1} = v_n + a_n \Delta t.$$ 

Note that everything on the right-hand side of the above equation is known, so we can compute $v_{n+1}$ using this equation. Decomposing into $x$- and $y$-components, we have

$$v_{x,n+1} = v_{x,n} + a_{x,n} \Delta t,$$

$$v_{y,n+1} = v_{y,n} + a_{y,n} \Delta t.$$ 

4. Derive analogous expressions to (2) and (3) for $x_{n+1}$ and $y_{n+1}$.

$$x_{n+1} = x_n + v_{x,n} \Delta t,$$

$$y_{n+1} = y_n + v_{y,n} \Delta t.$$ 

5. Implement the explicit Euler scheme you have developed in Microsoft Excel®. Start by creating user-defined input cells for the parameters $m$, $g$, $c$, $\Delta t$, and all the initial conditions (you can change the numerical values of these cells at will later). Next, create columns for $t$, $x$, $y$, $v_x$, $v_y$, $a_x$, and $a_y$ (the cells in these columns should reference the cells you created earlier). Be sure to label the appropriate units for every quantity. Finally, make a scatter plot of $y$ versus $x$—this is the trajectory of the object!
Application: Three-Pointer (Basketball)

You are standing at the three-point line of a regulation NBA basketball court. At your current position, you are 23.75 ft away (horizontally) from the basket, which is 10 ft above the court. Your teammate passes you the ball, which has a mass of 0.625 kg, and a drag parameter of approximately 0.015 kg/m in air. You toss the ball from a height of 2 m above the court at an angle of 45° with respect to the horizontal.

6. How fast must you throw the ball in order to make the basket (and score three points)?

*Roughly 9.78 m/s*

7. Would you have made the basket had you not accounted for air resistance?

*No. Had we not accounted for air resistance, we would have thrown the ball at 9.1 m/s, and the ball would have fallen short of the basket.*