Housekeeping

- Thurs: ME 25
- Discussion on Friday still on
- No class Friday
- No office hours last Sun of break
Slide 4

• Cables
  → when used to carry loads, weight is negligible (usually small compared to load)
  → if used for non-structural purposes (e.g. electrical wires) should include weight

• Assume
  • Perfectly flexible ⇒ No bending tension only
  • Inextensible ⇒ No change in length (geometry)

• Rigid body + we can use statics
We will consider 3 scenarios:

1. Concentrated loads
2. Distributed loads
3. Under its own weight
Cable Subjected to a Distributed Load
Draw FBD on slide

- has length $\Delta S$
- Forces changes along length ($w$ is constant)
- Dist load $\Rightarrow$ resultant $= w(x) \Delta x$
  - acts some $\gamma$ of $\Delta x$

$\Sigma F_x : (T + \Delta T) \cos(\theta + \Delta \theta) - T \cos \theta = 0$

$\Sigma F_y : (T + \Delta T) \sin(\theta + \Delta \theta) - w(x) \Delta x - T \sin \theta = 0$
\[ EM_0: \ w(x) \Delta x \cdot k(\Delta x) - T \cos \theta \Delta y + T \sin \theta \Delta x = 0 \]

Divide all \[ \Delta x \] by \[ \Delta x \], take \[ \lim \Delta x \to 0 \]

If \[ \Delta x \to 0 \] then \[ \Delta y \to 0 \]
\[ \Delta \theta \to 0 \]
\[ \Delta T \to 0 \]

From \[ EF_x: \]
\[ \frac{d}{dx} (T \cos \theta) = 0 \]

Integrate \[ T \cos \theta = \text{constant} = F_H(1) \]

\[ F_H \] is horizontal tensile force at any point along the cable

From \[ EF_y: \]
\[ \frac{d}{dx} (T \sin \theta) - w(x) = 0 \]

Integrate \[ T \sin \theta = \int w(x) \, dx \]

Divide by \[ (1) \]
\[
\frac{T \sin \theta}{T \cos \theta} = \frac{\int w(x) \, dx}{F_h}
\]
\[
\tan \theta = \frac{1}{F_h} \int w(x) \, dx \quad (2)
\]

From EM:
\[
\frac{dy}{dx} = \tan \theta \quad (3)
\]

\[
\tan \theta = \frac{dy}{dx} = \frac{1}{F_h} \int w(x) \, dx
\]

Integrate again:
\[
y = \frac{1}{F_h} \int \left( \int w(x) \, dx \right) \, dx
\]

\[
\text{curve for the cable (vertical position)}
\]
The cable of a suspension bridge supports half of the uniform road surface between the two towers at A and B. If this distributed loading is \( w_o \), determine the maximum force developed in the cable and the cable’s required length. The span \( L \) and sag \( h \) are known.
Approach:
Force \( \Rightarrow \) force will be a \( f(\Theta) \)

From (1) \( F_H = T \cos \Theta \)

\[ T = \frac{F_H}{\cos \Theta} \]

What range of \( \Theta \) will we consider?

By symmetry \( \Rightarrow 0 < \Theta < \frac{\pi}{2} \)

\( \cos 0 = 1 \)
\( \cos \frac{\pi}{2} = 0.9 \)

\( T \uparrow \text{ as } \Theta \uparrow \); \( T_{\max} \) occurs at \( B \)

What is \( \Theta_B \)? What is \( F_H \)?

\( \Rightarrow \text{Requires shape of cable} \)
Recall, \( y = \frac{1}{F_H} \int \int (\int w(x) \, dx) \, dx \)

Here \( w(x) = w_0 \)

\( \therefore y = \frac{1}{F_H} \int (\int w_0 \, dx) \, dx \)

\( \int w_0 \, dx = w_0 x + c_1 \)

\( y = \frac{1}{F_H} \int (w_0 x + c_1) \, dx \)

\( y = \frac{1}{F_H} \left( \frac{1}{2} w_0 x^2 + c_1 x + c_2 \right) \)

at \( x = 0 \quad y = 0 \)

\( 0 = \frac{1}{F_H} (0 + 0 + c_2) \)

\( \therefore c_2 = 0 \)
\[ \frac{dy}{dx} = 0 \quad \text{at} \quad x = 0 \]
\[ y' = \frac{1}{F_H} \left( \frac{1}{2} w_o \right) 2x + C_1 \]
\[ 0 = \frac{1}{F_H} (0 + C_1) \quad \therefore C_1 = 0 \]

Shape of cable:
\[ y = \frac{1}{F_H} \left( \frac{1}{2} w_o \right) x^2 \]

Recall \( \tan \Theta = \frac{dy}{dx} \) \( \leftarrow \) we want this at \( B \) \( (x = \frac{L}{2}) \)

\[ y = \frac{w_0}{2F_H} x^2 \]

\[ y' = \frac{w_0}{F_H} x = \frac{w_0 L}{2F_H} = \tan \Theta \]

\[ \Theta_{\max} = \Theta_B = \tan^{-1} \left( \frac{w_0 L}{2F_H} \right) \]
\[ T_{\text{max}} = \frac{F_{\text{H}}}{\cos \Theta_{\text{max}}} \]

Solve for \( F_{\text{H}} \)

\[ F_{\text{H}} = \frac{W_0 x^2}{2y} \]

when \( x = \frac{L}{2} \) \( y = h \)

\[ F_{\text{H}} = \frac{W_0 L^2}{2(4)(2)(h)} = \frac{W_0 L^2}{8h} \]

\[ \Theta_{\text{max}} = \tan^{-1} \left( \frac{\frac{W_0 L}{8h}}{\frac{x}{2W_0 L}} \right) = \tan^{-1} \left( \frac{4h}{L} \right) \]

\[ \text{Length} \]

\[ L = 2 \int_{0}^{\frac{L}{2}} ds \]

\[ ds = \sqrt{dx^2 + dy^2} \]

\[ ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \]
$$y' = \frac{w_0}{F_H} \chi = \frac{w_0 Bh}{k_0 L^2} \chi = \frac{Bh}{L^2} \chi$$

$$ds = \sqrt{1 + \left( \frac{Bh}{L^2} \chi \right)^2}$$

$$L = 2 \left[ \int_0^{\chi/2} \left[ 1 + \left( \frac{Bh}{L^2} \chi \right)^2 \right]^{1/2} d\chi \right]$$

... use an integral table or expand and solve
Cable Subjected to its own weight

\[ w = w(s) \]
Draw FBD on slide

- has length \( \Delta s \)
- Forces changes along length (\( w \) is constant)
- Dist load \( \Rightarrow \) resultant = \( w(s) \Delta s \)
  - acts some \( \theta \% \) of \( \Delta x \)

Apply equations of equilibrium:
\[
T \cos \theta = F_{\parallel} \\
T \sin \theta = \int w(s) \, ds \\
\frac{dy}{dx} = \frac{1}{F_{\parallel}} \int w(s) \, ds \\
\text{Note that} \\
ds = \sqrt{dx^2 + dy^2} \\
\text{Rearrange:} \quad \frac{dy}{dx} = \frac{1}{\sqrt{(dx/dx)^2 - 1}} \\
\text{Substitute into LHS above} \\
\sqrt{(dx/dx)^2 - 1} = \frac{1}{F_{\parallel}} \int w(s) \, ds\]
Rearrange:
\[ \frac{ds}{dx} = \left[ 1 + \frac{1}{F_H^2} \left( \int w(s) \, ds \right)^2 \right]^{1/2} \]

Separate variables
\[ x = \int \left[ 1 + \frac{1}{F_H^2} \left( \int w(s) \, ds \right)^2 \right]^{-1/2} \, ds \]

\[ \uparrow \]

relationship between x-position and the cable segment and horizontal force

Why do we allow cables to sag

1) ↓ sag ⇒ ↑ tension

2) Thermal expansion/contraction

↑ winter is the problem
The power transmission cable weighs 5lb/ft. If h=10ft, determine the length of the cable between the two towers.
**Slide 5**

**Approach:**
- Use symmetry and solve for $\frac{1}{2}$, then double.

**Diagram:**
- 15 ft
- $w_0 = \frac{5}{16} \text{ ft}$
- 150 ft

- Solve for $S$

$$X = \int \left[ 1 + \frac{1}{F_h^2} \left( \int w(s) ds \right)^2 \right]^{-\frac{1}{2}} ds$$

(!) Don't freak out... you can do this!

Let's start with the inner integral...

$$\int w(s) ds = \int w_0 ds.$$
Solve for $C_1$ at $s=0$, slope = 0

From way up above

\[ \frac{dy}{dx} = \frac{1}{F_H} \int w(s) \, ds \]

Our inner integral was:

\[ \int w(s) \, ds = \int w_0 \, ds = w_0 s + C_1 \]

\[ \frac{dy}{dx} = \frac{1}{F_H} (w_0 s + C_1) \]

at $s=0$, $\frac{dy}{dx} = 0$, $\therefore C_1 = 0$

\[ x = \int \frac{1}{\left[ 1 + \frac{1}{F_H^2} (w_0 s) \right]^2} \, ds \]

Variable substitution to make this easier...

let $u = \frac{1}{F_H} (w_0 s)$
\[ u = \frac{w_0}{F_H} s \]
\[ du = \frac{w_0}{F_H} ds \]

Substitute back in

\[ x = \int \frac{1}{\sqrt{1 + u^2}} \frac{F_H}{w_0} \, du \]
\[ x = \frac{F_H}{w_0} \int \frac{1}{\sqrt{1 + u^2}} \, du \]

Integration table

\[ x = \frac{F_H}{w_0} \left( \sinh^{-1} u + C_2 \right) \]

Substitute \( u \) back in

\[ x = \frac{F_H}{w_0} \left( \sinh^{-1} \left( \frac{w_0}{F_H} s \right) + C_2 \right) \]
Solve for $c_2$

At $x = 0$ \( S = 0 \)

\[ \sinh(0) = 0 \implies c_2 = 0 \]

\[ x = \frac{F_+}{w_0} \sinh^{-1} \left( \frac{w_0}{F_+} \right) S \]

Rearrange

\[ S = \frac{F_+}{w_0} \sinh \left( \frac{w_0}{F_+} x \right) \]

What is $F_+$?

Recall \( \frac{dy}{dx} = \frac{w_0}{F_+} S \)

\[ = \frac{w_0}{F_+} \cdot \frac{F_+}{w_0} \sinh \left( \frac{w_0}{F_+} x \right) \]

\[ \frac{dy}{dx} = \sinh \left( \frac{w_0}{F_+} x \right) \]

Integrate

\[ y = \frac{F_+}{w_0} \cosh \left( \frac{w_0}{F_+} x \right) + C_3 (?!) \]
Use BC \( x = 0 \) \( y = 0 \)

\[
\therefore \quad C_3 = -\frac{F_{ht}}{w_0}
\]

\[
\therefore \quad y = \frac{F_{ht}}{w_0} \left[ \cosh \left( \frac{w_0}{F_{ht}} x \right) - 1 \right]
\]

\( F_{ht} \) will be max at \( x = 150 \) ft, \( y = 15 \) ft

Solve for \( F_{ht} = 3762 \) lb

Now we can solve for \( x = 150 \) ft

\[
\delta = \frac{F_{ht}}{w_0} \sinh \left( \frac{w_0}{F_{ht}} x \right) = 151 \) ft

Then, length = 2\( \delta \)

\[
= 302 \) ft\]