To do ...

- HW 22 due **Today**
- HW 23(PL) due **Sunday**

- WA9 due **Sunday April 3**

- Quiz 5 (TAM 210 and 211) Tues April 5 to Sat April 9  
  * Sign up!

- Last TAM 210 class (ever!) **Friday April 8**
Rapid Refresh ...

- i
- <3
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- i
- c
- k
- e
- r
The composite method for determining the location of the center of gravity of a composite body requires ______.

A) Simple arithmetic       B) Integration
C) Differentiation        D) All of the above.
iQ>clicker

Based on typical available centroid information, what are the minimum number of pieces to consider for determining the centroid of the area shown at the right?

A) 5  B) 3  C) 4  D) 2  E) 6
iQ>clicker

A storage box is tilted up to clean the rug underneath the box. It is tilted up by pulling the handle C, with edge A remaining on the ground. What is the maximum angle of tilt possible (measured between bottom AB and the ground) before the box tips over?

A) 30°  B) 45°  C) 60°  D) 90°
Second moment of area

Consider a beam.

Consider a plank

Q: Which is safer?

A) I

B) II

C) No difference
\begin{align*}
L_1 &= L_2 = L_3 \\
L_1 < L_2 < L_3 \\
\text{Compression} \\
\text{Tension} \\
L_2 &= L_0
\end{align*}

\[ \sigma = \text{Stress} = \frac{F}{A} \]
Second moment of area

**Moment of inertia** is the property of a deformable body that determines the moment needed to obtain a desired curvature about an axis.

Moment of inertia depends on the shape of the body and may be different around different axes of rotation.

- The moment of inertia of the area $A$ with respect to the $x$-axis is given by
  \[ I_x = \int y^2 \, dA \]

- The moment of inertia of the area $A$ with respect to the $y$-axis is given by
  \[ I_y = \int x^2 \, dA \]
Parallel axis theorem

- Often, the **moment of inertia** of an area is known for an axis passing through the **centroid**; e.g., $x'$ and $y'$:
- The moments around other axes can be computed from the known $I_x'$ and $I_y'$:

\[
I_x = \int_A (d^2 + 2dy' + y'^2) \, dA
\]

\[
I_y = \int_A x'^2 \, dA
\]

\[
I_x = \int_A x'^2 \, dA + I_{x'}
\]

\[
I_y = \int_A y'^2 \, dA + I_{y'}
\]

\[
I_y = \int_A y'^2 \, dA
\]

\[
\bar{y} = \frac{\int y \, dA}{\int dA} = 0
\]

\[
\int y' \, dA = 0
\]
\begin{align*}
  I_x &= I_{x'} + A \delta^2_y \\
  I_y &= I_{y'} + A \delta^2_x
\end{align*}
Determine the moment of inertia for the rectangular area shown w.r.t. (a) the centroidal axis $x'$ and (b) the axis passing through the base of the rectangle $x_b$.

\[ A = h \frac{b}{2} \]

\[ I_{y'} = I_{x'} \alpha b \]

\[ I_{x'} \alpha h^3 \]

\[ I_{x'} = \frac{1}{12} bh^3 \]

\[ I_{y'} = \int x'^2 dA = \int x'^2 h \, dx' = \frac{1}{3} x'^3 \bigg|_{-\frac{b}{2}}^{\frac{b}{2}} = \frac{1}{12} bh^3 \]

**NOW ABOUT $x_b$, USE THE PARALLEL AXIS THEOREM**

\[ I_{x_b} = I_{x'} + Ad^2 = \frac{1}{12} bh^3 + \left( bh \frac{h}{2} \right)^2 \]

\[ = \frac{1}{12} bh^3 + \frac{bh^3}{4} = \frac{1}{3} bh^3 \]
USE the tables.
PAY attention to axis!
Moment of inertia of composite

- If individual bodies making up a composite body have individual areas $A$ and moments of inertia $I$ computed through their centroids, then the composite area and moment of inertia is a sum of the individual component contributions.
- This requires the parallel axis theorem
- Remember:
  - The position of the centroid of each component must be defined with respect to the same origin.
  - It is allowed to consider negative areas in these expressions. Negative areas correspond to holes/missing area. This is the one occasion to have negative moment of inertia.

\[ M_{0} \text{I always positive} \quad \int y^{2} \text{d}A \]
Applications

Many structural members like beams and columns have cross sectional shapes like an I, H, C, etc..

Why do they usually not have solid rectangular, square, or circular cross sectional areas?

What primary property of these members influences design decisions?

\[ I_x = \text{moment of inertia about the x-axis} \]
Applications

Many structural members are made of tubes rather than solid squares or rounds. Why?

This section of the book covers some parameters of the cross sectional area that influence the designer’s selection.
Mass Moment of Inertia

- **Mass moment of inertia** is the mass property of a rigid body that determines the torque $T$ needed for a desired angular acceleration ($\alpha$) about an axis of rotation.
- A larger mass moment of inertia around a given axis requires more torque to increase the rotation, or to stop the rotation, of a body about that axis.
- Mass moment of inertia depends on the shape and density of the body and is different around different axes of rotation.

Torque-acceleration relation:

\[ T = I \alpha \]

\[ F = ma \]

- Disk
- Hole
- Fly wheel

\[ R_H = \left( \frac{80}{100} \right) R_D \]

Calculate the mass of each element:

\[ M_D = \rho V = \rho \pi r^2 t = \rho \pi \frac{R_D^2}{2} t \]
\[ M_H = \frac{4}{3} \pi R_H^3 \]
\[ M_f = M_D - M_H = \frac{4}{3} \pi t \left( R_D^2 - R_H^2 \right) = \frac{4}{3} \pi t R_D^2 \left( 1 - \left( \frac{R_H}{R_D} \right)^2 \right) \]
\[ \frac{M_f}{M_D} = \frac{\frac{4}{3} \pi R_D^2 t \left( 1 - \left( \frac{R_H}{R_D} \right)^2 \right)}{\frac{4}{3} \pi R_D^2 t} = \left( 1 - \frac{36}{100} \right) = 0.64 = 0.36 \]
\[ M_f = (36\%) M_D \]
great!!

But what about \( I_m \)?

\[ I = \frac{1}{2} m r^2 \]
\[ I_f = I_D - I_H = \frac{1}{2} M_D R_D^2 - \frac{1}{2} M_H R_H^2 \]
\[ I_f = \frac{1}{2} M_D R_D^2 \left( 1 - \left( \frac{M_H}{M_D} \right) \left( \frac{R_H}{R_D} \right)^2 \right) \]
\[ I_f = \frac{1}{2} M_D R_D^2 \left( 1 - \left( \frac{\frac{4}{3} \pi t R_D^2 \left( 1 - \left( \frac{R_H}{R_D} \right)^2 \right)}{\frac{4}{3} \pi t R_D^2} \right) \left( \frac{R_H}{R_D} \right)^2 \right) = \frac{1}{2} M_D R_D^2 \left( 1 - \left( \frac{R_H}{R_D} \right)^3 \right) \]
\[ I_f = I_D \left( 1 - \left( \frac{3}{10} \right)^3 \right) = 49\% I_D \tag{not bad.} \]

\[ \text{Cut mass by 64\%, Cut } I_m \text{ by 51\%} \]
Cut mass by 64%, cut $I_m$ by 51%
Determine the moment of inertia for the shaded area about the x axis.

\[ I_x = \int_A y^2 dA \]

Use parallel axis theorem.

\[ I_x = I_{x, \text{par}} - I_{x, 0} \]

\[ I_{x, \text{par}} = I_x + Ad^2 = \frac{1}{12} bh^3 + A d^2 = (\frac{1}{12} \times 100 \times 75)^3 + (100 \times 75 \times 25)^2 \]

\[ I_{x, 0} = I_x + Ad^2 = \frac{1}{4} \pi r^4 + A y^2 = \frac{\pi}{4} (25)^4 + (100 \times 75)^2 \]

Units?

\[ \text{mm}^4 \]

\[ I_x = I_{x, \text{par}} - I_{x, 0} = 112.5 \times 10^4 - 1.4 \times 10^6 = 101.6 \times 10^6 \text{ mm}^4 \]
Centroid position of the area below is given by

\[ A_{\text{total}} \bar{Y} = \sum_i A_i \bar{y}_i \]

\[ \bar{Y} = \frac{4t^2 (3.5t) + 6t^2 (1.5t)}{4t^2 + 6t^2} = \frac{23t}{10} \]

Find the moment of inertia:

**For section I:**

- \( d_1 = \frac{7}{2}t - \frac{23}{10}t = 1.2t \)
- \( A_1 = 4t^2 \)
- \( I_1 = \frac{1}{12} (4t^2)(t^3) = \frac{1}{3}t^4 \)

The moment about \( x' \) is then

\[ I_{x'} = I_{x_1} + I_{x_2} = (I_1 + A_1 d_1^2) + (I_2 + A_2 d_2^2) \]

\[ I_{x'} = \frac{1}{3}t^4 + \left(4t^2 \left(\frac{12}{10}t\right)^2\right) + \frac{84}{12}t^4 + (6t^2 \left(\frac{8}{10}t\right)^2) \]

\[ I_{x'} = 14.4t^4 \]
Determine the moment of inertia for the cross-sectional area about the \(x\) and \(y\) centroidal axes.

Divide the section into rectangular areas.

The MoI about \(x\) is:

For \(A\) and \(D\),

\[
I_x = I_x' + Ad_y^2
\]

\[
I_x = \frac{1}{12} \left(100 \times 300^3 + (100 \times 300 \times 200)^2\right) = 1.425 \times 10^9 \text{ mm}^4
\]

For \(B\),

\[
I_x = \frac{1}{12} (600 \times 100)^3 = 0.05 \times 10^9 \text{ mm}^4
\]

The MoI about \(x\) is then

\[
I_x = 2 \left[ 1.425(10^9) \right] + 0.05(10^9) = 2.9(10^9) \text{ mm}^4
\]

The MoI about \(y\) is:

For \(A\) and \(D\),

\[
I_y = I_y' + Ad_x^2 = \frac{1}{12} \left(300 \times 100^3 + (100 \times 300 \times 250)^2\right) = 1.9 \times 10^9 \text{ mm}^4
\]
And for B
\[ I_y = \frac{1}{12} (100 \times 600)^3 = 1.8 \times 10^9 \text{ mm}^4 \]

The total MoI about y is

\[ I_y = 2 \times 1.9 \times 5 \times 10^9 + 1.8 \times 10^9 = 5.6 \times 10^9 \text{ mm}^4 \]

Q: Can you explain why \( I_y > I_x \)?
<table>
<thead>
<tr>
<th>Designation</th>
<th>Area $\text{in}^2$</th>
<th>Depth $\text{in}$</th>
<th>Width $\text{in}$</th>
<th>Axis $X-X$</th>
<th>Axis $Y-Y$</th>
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</table>

Angles:
- L5 x 6 x 1 | 11.0 | 35.4 | 1.79 | 1.86 |
- L4 x 4 x 1 | 3.75 | 5.22 | 1.21 | 1.18 |
- L3 x 3 x 1 | 1.44 | 1.33 | 0.92 | 0.83 |
- L5 x 4 x 1 | 4.75 | 17.3 | 1.91 | 1.96 |
- L3 x 3 x 1 | 3.75 | 9.43 | 1.38 | 1.74 |
- L3 x 2 x 1 | 1.19 | 1.00 | 0.95 | 0.86 |

**USE the Charts!**
<table>
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<tr>
<th>Designation</th>
<th>Area $\text{mm}^2$</th>
<th>Depth $\text{mm}$</th>
<th>Width $\text{mm}$</th>
<th>Axis X-X $10^6\text{mm}^4$</th>
<th>$\bar{y}$</th>
<th>Axis Y-Y $10^3\text{mm}^3$</th>
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</table>
Two channels are welded to a rolled W section as shown. Determine the moments of inertia of the combined section with respect to the centroidal x and y axes.