Lecture 18

Responses in thick plates.

Standard Procedure: Re-solve the PDE, but with inhomogeneous BCs related to applied surface loads. Or, if the loads are internal (maybe from a crack growth) re-solve the PDE with traction free surface conditions but with a discontinuity in stress at the depth of the load. We'll first examine the former case. Later we'll examine a very different approach that is arguably much simpler conceptually and algebraically, and as easily applied to the buried source problem as the surface source problem.

Consider the concentrated step line load on the top surface, uniform in the $z$ (into the paper) direction.

Our equations are identical to those on page 139 except that now the traction BCs are

$$\sigma_{yy}(y = h) = \Theta(t) \delta(x)$$

$$\sigma_{yy}(y = -h) = \sigma_{xy}(y = h) = \sigma_{xy}(y = -h) = 0$$

We do our familiar Helmholtz decomposition

$$u_x = \partial_x \Phi + \partial_y H; \quad u_y = \partial_y \Phi - \partial_x H$$

$$\sigma_{yy} = (\lambda + 2\mu) \nabla^2 \Phi - 2\mu \partial_x^2 \Phi - 2\mu \partial_x \partial_y H; \quad \sigma_{xy} = 2\mu \partial_x \partial_y \Phi + \mu (\partial_y^2 - \partial_x^2) H$$

We do our familiar double FT and solve the resulting ODEs to get

$$\tilde{\Phi}(y; \omega, \xi) = A \sin \alpha y + B \cos \alpha y; \quad \tilde{H}(y; \omega, \xi) = D \cos \beta y + C \sin \beta y$$

A and D describe antisymmetric waves; B and C describe symmetric waves. Let us treat only the symmetric waves (and modify our load so that there is an upward step load on the top and a simultaneous downward one on the bottom, so that the loading is symmetric too.)

$$\tilde{\Phi} = B \cos \alpha y; \quad \tilde{H} = C \sin \beta y$$
The displacements are

\[ \tilde{u}_x = -i\xi B \cos \alpha y + \beta C \cos \beta y; \quad \tilde{u}_y = -\alpha B \sin \alpha y + i\xi C \sin \beta y \]

And the stresses are

\[ \tilde{\sigma}_{yy} = -(\omega^2 / c_L^2)(\lambda + 2\mu)B \cos \alpha y + 2\mu \xi^2 B \cos \alpha y + 2i\xi \mu \beta C \cos \beta y \]
\[ = \mu(\xi^2 - \beta^2)B \cos \alpha y + 2i\xi \mu \beta C \cos \beta y \]
\[ \tilde{\sigma}_{xy} = 2i\xi \alpha \mu B \sin \alpha y + \mu(-\beta^2 + \xi^2)C \sin \beta y \]

We wish to set these to \((1/i\omega)\) and zero respectively, at \(y = +h\). This automatically sets them to the same values at \(y = -h\).

Thus

\[
\begin{bmatrix}
(\xi^2 - \beta^2) \cos \alpha h & 2i\xi \beta \cos \beta h \\
2i\xi \alpha \sin \alpha h & (\xi^2 - \beta^2) \sin \beta h
\end{bmatrix}
\begin{bmatrix}
B \\
C
\end{bmatrix} = \frac{1}{i\mu \omega}
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

The SV wave response is given by

\[ C \sin \beta y = \frac{-(2\alpha \xi / \mu \omega) \sin \alpha h \sin \beta y}{(\xi^2 - \beta^2)^2 \sin \beta h \cos \alpha h + 4\xi^2 \alpha \beta \cos \beta h \sin \alpha h} \]

There is a similar expression for \(B\cos \alpha y\). The procedure would be even more complicated if the source had acted in the interior. Then you'd find that \(H\) and \(\Phi\) satisfied ODEs in \(y\) with an inhomogeneous term, possibly with jump conditions in the interior. As alluded to above, I will shortly be giving a very different – and I think simpler – way to construct the transformed displacement response, equally adaptable to surface and to buried sources.

Notice that, once again, the dispersion relation is in the denominator. Thus when we perform the inverse spatial transform, we can opt to evaluate it by summing the contributions from all the poles, i.e, the zeros of the denominator, i.e, the points \(\xi(\omega)\) of the dispersion relation.

For real \(\omega\), when you do the integral over \(\xi\), you find that the \(\xi\) that satisfy the dispersion relation are both real (there are a finite number of real \(\xi\)) and complex (an infinite number, most of them imaginary). So to do this requires that you know the complex branches of the dispersion relations (not plotted above) as well as the real branches. You'd be left with a temporal inverse FT to perform numerically, of a sum over an infinite number of branches, most of them complex. A finite \(x\), however, the contributions from branches
with large imaginary $\xi$ will be negligible; so the sum can be truncated. At $x = 0$, you need all terms.

Alternatively you could opt to first perform the inverse temporal transform, and pick up contributions at all the poles $\omega(\xi)$. These roots $\omega$ are all real, so the algebra and bookkeeping is a bit simpler. For any fixed $\xi$ there are an infinite number of roots $\omega$. But you can usually justify truncating that sum by confining interest to finite frequencies. You'd be left with a spatial inverse transform to perform numerically, an integration over a sum of branch contributions.

Depending on your distances and frequencies of interest it may be efficient to do the inverse spatial and temporal transforms as described above. At high bandwidths, you need lots of branches, so it can be burdensome. However, the Cagniard procedure gives expressions that cover all frequencies and are essentially exact. It is very cumbersome at large $r >> 2h$ (for reasons discussed below) but is often carried out for short distances $r$ - no more than several thicknesses from the source. See Knopoff (J Appl Phys 24, 661-670 (1958), or Ahmet Ceranoglu and YH Pao J. Applied Mechs (1980.) I won't go into this procedure in much detail. But I will outline the procedure: Here is the sort of expression that we have for the inverse spatial transform needed to construct the SV portion of $\tilde{u}(\omega)$:

$$\int \xi^2 \, d\xi J_0(\xi r) \frac{(-2\alpha / \mu \omega) \sin \beta y \sin \alpha h}{(\xi^2 - \beta^2)^2 \sin \beta h \cos \alpha h + 4\xi^2 \alpha \beta \cos \beta h \sin \alpha h}$$

We can see the main features best in the simple case (Knopoff) of epicenter, $y = -h$, $r = 0$ for which $J = 1$.

$$\int \xi \, d\xi \frac{(2\alpha / \mu \omega) \sin \beta h \sin \alpha h}{(\xi^2 - \beta^2)^2 \sin \beta h \cos \alpha h + 4\xi^2 \alpha \beta \cos \beta h \sin \alpha h}$$

Cagniard requires an integral like $\int d\xi \, f(\xi) \exp(-i\alpha y)$ (in which $f$ is a homogeneous function of $\omega$ and $\xi$, that can be written like $\omega^p \, g(\xi/\omega)$ ) One then substitutes $\alpha y = \omega \tau$ and deforms the integration path so that one integrates over real $\tau$. The resulting expression looks like a forward temporal FT, and can then be inverted by inspection to recover $u$ in the time domain. The problem here is that the above integral is NOT of the Cagniard form.

It can, however, be turned into a sum of Cagniard-like forms, each with the desired exponential like factor $\exp(-i\alpha y)$. By expanding the trig functions (eg.: $\cos x = (\exp(ix)+\exp(-ix))/2$ ), and writing the inverse of the denominator in a power series in powers of $\exp(i\alpha h)$ etcetera, one can rewrite the above integrand as a sum of terms like $\exp(-2im\beta h)$ corresponding to $n$ trips through the thickness as a P wave and $m$ trips as an S wave. Each such term is then in Cagniard form and can be inverted exactly. The result is an expression for the response that is a sum of contributions each identifiable with a ray (a so-called "generalized ray.") Each ray has an arrival time (corresponding to delay by traveling through the thicknesses many times) and an amplitude (that must be a product of
many reflection coefficients, some of them mode conversions) The result of this algebraic manipulation is to turn the above into a sum (over all \( n,m \)) of expressions that look like

\[
\int d\xi f_{nm}(\xi/\omega) \exp(-2in\alpha - 2im\beta)
\]

for which each term can be subjected to the Cagniard method. That term corresponds to a ray that traveled through the \( 2h \) thickness \( n \) times as a \( P \) wave and \( m \) times as an \( S \) wave. (If \( n + m \) is even, and our detection point \( y \) is at \( y = -h \) while the source was at \( +h \), then it is a ray that traveled from the fake source and can be neglected.)

The difficulty is two-fold 1) the algebra is intense, see Ceranoglu and 2) there are many rays. At large times one needs many rays. At large distance \( r \) one has many rays that arrive even at times only shortly after the first arrival. Due to mode conversion, the number of rays rises exponentially with time of interest. The advantage is that the result is exact, with few numerical integration accuracy concerns (at \( r \neq 0 \) each ray calculation requires an integral over a finite range of a dummy variable so there is some concern). Nor is there any need to truncate sums over branches in adhoc manners.

One nice thing about this representation is that we need not go back and consider the antisymmetric waves; in the ray sum it will be clear which rays come from the fake source; they can be neglected. Furthermore in the theoretical response \( u(t) \) one can identify the ray processes that contribute to each feature.

If you want to know more about this approach, the best place to start is the paper by Knopoff. My opinion is that such methods are these days less attractive than they were before. For most problems I find that modern computation and brute force integration almost always give adequate answers with less conceptual or algebraic effort.

We'll come back to responses in plates later, after discussing normal modes of finite bodies

Circular rods. Guided Pochammer waves. (Graff 8.2)

We use circular cylindrical coordinates \( r,z,\theta \). Our displacement field has components in the \( r \) \( z \) and \( \theta \) directions. Our Helmholtz decomposition is

\[
\vec{u} = \nabla \Phi + \nabla \times \vec{H}
\]

\( \vec{H} \) has three components, but we only need it to provide two degrees of freedom. Graff inserts a gauge condition on \( \vec{H} \) that its divergence is specified somehow. It is I believe more common, but maybe equivalent(?), to take \( \vec{H} \) in the form
\[
\hat{H} = k \hat{H}_z + a \nabla \times \hat{k} H_+ \\
= \hat{k} \hat{H}_z + (a \hat{r} / r) \partial_\theta H_+ - (a \hat{\theta}) \partial_r H_+
\]
(The factor of \(a\) is inserted for dimensional regularity)

so that

\[
\vec{u} = \nabla \Phi + \nabla \times \hat{k} \hat{H}_z + a \nabla \times \nabla \times \hat{k} H_+
\]

The three potentials satisfy simple scalar wave equations (simple because \(\hat{k}\) is a constant)

\[
c_L^2 \nabla^2 \Phi(r, z, \theta, t) = \ddot{\Phi} ; \quad c_T^2 \nabla^2 H_z(r, z, \theta, t) = \ddot{H}_z ; \quad c_T^2 \nabla^2 H_+(r, z, \theta, t) = \ddot{H}_+
\]

We take all fields to have \(z, t\) and \(\theta\) dependence of the form \(\exp(i \omega t - iq z + in \theta)\) times a function of \(r\), as informed by the translation invariance in \(t\) and \(z\) and rotational invariance in \(\theta\). Alternatively, you may think of \(\omega, q\) as FT variables and \(n\) as a Fourier series index. The axisymmetry and \(z\)-translation invariance assure that the different \(q\) and \(n\) do not mix.

Then suppressing the exponentials, the ODEs for the \(r\)-dependence of the displacement potentials (with \(\theta, t, z\) dependence suppressed) are simply (Recall \(\nabla^2\) in cylindrical coordinates is \(\partial_r^2 + (1/r) \partial_r + \partial_\theta^2 + 1/r^2 \partial_\theta^2\))

\[
c_L^2 \{\partial_r^2 + (1/r) \partial_r - q^2 - n^2 / r^2\} \Phi(r) = -\omega^2 \Phi(r) ; \quad c_T^2 \{\partial_r^2 + (1/r) \partial_r - q^2 - n^2 / r^2\} H_z(r) = -\omega^2 H_z(r) ; \quad c_T^2 \{\partial_r^2 + (1/r) \partial_r - q^2 - n^2 / r^2\} H_+(r) = -\omega^2 H_+(r)
\]

The solution to any one of these is a Bessel function \(J_n\) (and/or a Neumann function \(Y_n\) but regularity at the origin precludes the use of them unless there is an internal surface, as there will be if the rod has a hollow core—like a pipe.)

\[
\Phi(r) = AJ_n(\alpha r) ; \quad H_z(r) = BJ_n(\beta r) ; \quad H_+(r) = CJ_n(\beta r)
\]

where \(\alpha = \sqrt{\omega^2 / c_L^2 - q^2}\) and \(\beta = \sqrt{\omega^2 / c_T^2 - q^2}\) are the radial wave numbers

We must choose \(A, B, C\) so that the three tractions at the surface at \(r = a\) vanish.
It is clear, even before doing the algebra, that each component of the displacement $u$ is some linear combination of $A$, $B$, and $C$. Similarly each component of the traction at the surface is a linear combination of these three numbers $A$, $B$, $C$. The coefficients of those linear combinations will include Bessel functions and their derivatives. After doing that algebra, one will be left with a matrix condition of the form $[M] \{A \ B \ C\}^T = \{0 \ 0 \ 0\}^T$ where $M$ is a $3 \times 3$ matrix whose elements include Bessel functions and depend on $q$ and $\omega$ and $n$. A non trivial solution requires $\det M = 0$. This is an implicit function relating $\omega$ to $q$ for specified fixed $n$. It will, as in the case of the plate, have an infinite number of roots $\omega$. For fixed $n$, one finds a multi-branched relation. Graff on p 473,4 gives examples.

The displacement is (using the identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$)

$$\vec{u} = \vec{\nabla} \Phi + \vec{\nabla} \times \{\hat{k}H_z\} + a\vec{\nabla} \times \nabla \times \{\hat{k}H_z\}$$

$$= \vec{\nabla} \Phi + \vec{\nabla} \times \{\hat{k}H_z\} + a\{\nabla \partial_z H_+\} - a\hat{k}\nabla^2 H_+$$

$$= \vec{\nabla} \Phi + (\hat{r} / r)\partial_\theta H_z - \hat{\theta} \partial_r H_z + a\{\vec{\nabla} \partial_z H_+\} - a\hat{k}\nabla^2 H_+$$

Let us continue the analysis, but simplify our life by choosing $n = 0$ (which implies $\partial_\theta = 0$). Graff p 470. Then, after dropping $\partial_\theta$,

$$\vec{u} = \hat{r}[\partial_r \Phi + a\partial_z \partial_r H_+] + \hat{k}[\partial_z \Phi + a\partial_z^2 H_+ + a(\omega^2 / c_T^2)H_+] - \hat{\theta} \partial_r H_z$$

The needed strains are (see Graff p 465) (and continuing to take $\partial_\theta = 0$)

$$\varepsilon_{rr} = \partial_r u_r; \quad \varepsilon_{\theta\theta} = u_r / r; \quad \varepsilon_{zz} = \partial_z u_z$$

$$\varepsilon_{r\theta} = (1 / 2)(\partial_r u_\theta - u_\theta / r); \quad \varepsilon_{rz} = (1 / 2)(\partial_r u_z + \partial_z u_r)$$

or,

$$\varepsilon_{rr} = \partial_r^2 (\Phi + a\partial_z H_+); \quad \varepsilon_{r\theta} = -(1 / 2)(\partial_r - 1 / r)\partial_\theta H_z; \quad \varepsilon_{rz} = \partial_r \partial_z \Phi + a\partial_z \partial_z^2 H_+ + \frac{a\omega^2}{2c_T^2}\partial_r H_z$$

The tractions are

$$\sigma_{rr} = \lambda \vec{\nabla} \cdot \vec{u} + 2\mu \varepsilon_{rr} = \lambda \vec{\nabla}^2 \Phi + 2\mu \varepsilon_{rr}; \quad \sigma_{rz} = 2\mu \varepsilon_{rz}; \quad \sigma_{r\theta} = 2\mu \varepsilon_{r\theta}$$

or,
\[ \sigma_{rr} = -\lambda \frac{\omega^2}{c_L^2} \Phi + 2\mu \partial_r^2 \Phi + 2\mu \partial_z \partial_r^2 H_+ \]
\[ \sigma_{rz} = 2\mu (\partial_r \partial_z \Phi + a \partial_r \partial_z^2 H_+ + \frac{a\omega^2}{2c_T^2} \partial_r H_+) \]
\[ \sigma_{\theta r} = -\mu (1/2)(\partial_r - 1/r) \partial_r H \]

In terms of the ABC and evaluated at \( r = a \), the tractions are
\[ \sigma_{rr} = -\lambda \frac{\omega^2}{c_L^2} AJ_o'(\alpha a) + 2\mu \alpha^2 AJ_o''(\alpha a) + 2\mu a(-iq)\beta^2 BJ_o''(\beta a) = 0 \]
\[ \sigma_{rz} = 2\mu [-iq\alpha AJ_o'(\alpha a) + a\beta(\partial_z \Phi + \frac{a\omega^2}{2c_T^2})] = 0 \]
\[ \sigma_{\theta r} = -\mu [\beta^2 J_o''(\beta a) - (\beta/r) J_o'(\beta a)] C = 0 \]

The equation for \( C \) decouples from the others (This only happens at \( n = 0 \)). These are torsional waves, with displacement purely in the \( \theta \) direction.

\[ \ddot{u} = -\hat{\theta} C \beta J_o'(\beta r) \]

Their dispersion relation is
\[ \beta^2 J_o''(\beta a) - (\beta/r) J_o'(\beta a) = 0 \]

It has roots at \( \beta a = 0, 5.136, 8.417... \) separated by about \( \pi \). From this one can calculate the cutoff frequencies. Its branches look, qualitatively, like those of the SH waves in the plate.

The lowest torsional branch has \( \beta = 0 \) and is readily understood. It has \( \ddot{u} = \hat{\theta} r \) and is a simple twist.

The higher branches \( \beta > 0 \) have nodes on interior cylindrical radii; the direction of twist is different at different radii.
The other waves at $n = 0$ (called "longitudinal"—perhaps a bit misleadingly, because they include motion in $r$ as well as $z$ direction) are found by setting the determinant of the coefficients of $A$ and $B$ to zero. This results in a transcendental equation for $\omega$ as a function of $q$. When plotted, it looks qualitatively lots like the symmetric Rayleigh Lamb dispersion relation for plates (see Graff for plots).

For $n = 0$ one has decoupled torsional and 'longitudinal' branches. At very low frequency (long wavelength) these cases simplify into the torsional wave described above and to a simple bar wave (with speed $\sqrt{E/\rho}$) (see Graff's plots at $n = 0$).

For each $n > 0$, one has 3 fully coupled ABC types of waves and a complicated family of dispersion curves. Graff gives an example at $n = 1$ taken from Pao. At very low frequency the first branch becomes a familiar bending wave.

Guided waves in pipes are of special interest in engineering NDE; they permit remote interrogation of points on pipes that are not directly accessible to visual or other inspection. See the extensive work of Cawley and Lowe et al over the past couple of decades.