Elastic Waves HW 7 Solution

7.1 A half space is bounded by a rigid surface with boundary condition $u(x,y) = 0$. Consider displacement potentials of the form

$$\Phi = [A \exp(i\alpha y) + B \exp(-i\alpha y)] \exp(i\omega t - i\xi x)$$

$$H = [C \exp(i\beta y) + D \exp(-i\beta y)] \exp(i\omega t - i\xi x)$$

Find the 4 reflection coefficients as a function of $\xi, \omega$. P→P, S→S, P→S and S→P.

We use $u_x = \partial_x \Phi + \partial_y H$; $u_y = \partial_y \Phi - \partial_x H$ and deduce that, at the surface,

$$u_x = -i\xi(\alpha + \beta) + i\alpha(B - D)$$

$$u_y = i\alpha(\alpha - \beta) + i\xi(C + D)$$

On setting these to zero the condition may be combined into a matrix relation

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix}_{y=0} = \begin{bmatrix} -i\xi & i\beta \\ i\alpha & i\xi \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} + \begin{bmatrix} -i\xi & -i\beta \\ -i\alpha & i\xi \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} -\xi & \beta \\ \alpha & -\xi \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \xi & \beta \\ \alpha & \xi \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}$$

In the case $C=0$ $A=1$ (incident P wave) we deduce $\xi B + \beta D = -\xi$ and $\alpha B - \xi D = \alpha$. These may be solved: $D= -2\alpha\xi/(\xi^2 + \alpha\beta)$; $B = (-\xi^2 + \alpha\beta)/(\xi^2 + \alpha\beta)$.

In the case $A=0$ $C=1$ (incident SV wave) we deduce $\xi B + \beta D = \beta$ and $\alpha B - \xi D = \xi$. These may be solved: $D = (-\xi^2 + \alpha\beta)/(\xi^2 + \alpha\beta)$; $B = 2\beta\xi/(\xi^2 + \alpha\beta)$.

For an incident P wave with no incident SV wave (i.e. $C=0$) show that your coefficients are such that energy is conserved.

We ask if (when $A = 1$, $D = -2\alpha\xi/(\xi^2 + \alpha\beta)$, $B = (-\xi^2 + \alpha\beta)/(\xi^2 + \alpha\beta)$)

$$|A|^2 \cos\theta_t/c_L = |B|^2 \cos\theta_t/c_L + |D|^2 \cos\theta_T/c_T.$$?

We notice that $\cos\theta_L/c_L$ is $\alpha/\omega$ and $\cos\theta_T/c_T$ is $\beta/\omega$ and restate our query: Does

$$|A|^2 \alpha = |B|^2 \alpha + |D|^2 \beta?$$

A little bit of further algebra establishes that this is indeed satisfied. The key is to recognize that $(-\xi^2 + \alpha\beta)^2 + 4\xi^2 \alpha\beta = (\xi^2 + \alpha\beta)^2$. We also ought recognize that all the coefficients are real, so we can take ordinary squares and not worry about absolute values $|\cdot|^2$.

For an incident SV wave with no incident P wave (i.e. $A=0$) and angles of incidence less than critical, show that your coefficients are such that energy is conserved.

Again we do not worry about absolute values because all coefficients are real. We ask if (when $C=1$, $D = (-\xi^2 + \alpha\beta)/(\xi^2 + \alpha\beta)$; $B = 2\beta\xi/(\xi^2 + \alpha\beta)$)
\[ |C|^2 \cos \theta T/c_T = |B|^2 \cos \theta_L/c_L + |D|^2 \cos \theta_T/c_T. \]

I.e., does
\[ |C|^2 \beta = |B|^2 \alpha + |D|^2 \beta? \]

A little bit of further algebra establishes that this is indeed satisfied. The key is to again recognize that 
\[ (-\xi^2 + \alpha \beta)^2 + 4\xi^2 \alpha \beta = (\xi^2 + \alpha \beta)^2. \]

For an incident SV wave with no incident P wave (i.e. \( A = 0 \)) and angles of incidence greater than critical such that \( \alpha \) is negative imaginary, show that your coefficients are such that energy is conserved.

Now we need to know if, for \( C = 1, \ A = 0, \) with \( C = 1, \ D = (-\xi^2 + \alpha \beta)/(\xi^2 + \alpha \beta); \)
\( B = 2\beta \xi/(\xi^2 + \alpha \beta) \) at angles such that \( \alpha \) is imaginary, if
\[ |C|^2 \beta = |D|^2 \beta? \]

That is, is \( |D| = 1 \)? The value of \( B \) is irrelevant. We see that indeed \( |D| = 1 \), by noting that the numerator and denominator of \( D = (-\xi^2 + \alpha \beta)/(\xi^2 + \alpha \beta) \) are complex conjugates of each other.

Show that there is no surface wave solution, i.e, no solution with \( A=C=0, \alpha,\beta \) negative imaginary
(other than the trivial \( A=B=C=D=0 \)).

We seek a solution of
\[
\begin{pmatrix}
0 \\
0
\end{pmatrix} = \begin{pmatrix}
\xi & \beta \\
\alpha & -\xi
\end{pmatrix}\begin{pmatrix}
B \\
D
\end{pmatrix}
\]

A non trivial solution requires the det to vanish, i.e \( \xi^2 + \alpha \beta = 0 \) (in the regime in which \( \alpha,\beta \) are negative imaginary. i.e, \( \xi > \omega/cT \))
\[ \xi^2 - \sqrt{\xi^2 - \alpha \beta} = 0 \]

It should be apparent that there are no such solutions; the first term is clearly greater than the second term. We conclude that there are no surface waves.

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7.2] A problem using slowness surfaces. Two isotropic elastic half spaces are joined by a weld. The upper half space has \( c_L = 7 \) mm/msec, \( c_T = 4 \) mm/msec. The lower half space has \( c_L = 3.5 \) mm/msec and \( c_T = 2.5 \) mm/msec. An SV plane wave is incident upon the interface from below at an angle \( \theta \) from the normal. For 4 cases (\( \theta = 50^\circ, 40^\circ, 30^\circ \) and \( 15^\circ \)) determine what waves will propagate away from the interface by reflection or transmission, and what waves will evanesce. Find the angles from the normal of each of the outgoing propagating waves.
This may be done graphically. It may also be done purely using Snell's law. I'll do it that way (in order not to have to make digital slowness diagrams.)

The incident field has a horizontal slowness of $\sin \theta/2.5$ (The 4 values of this are 0.306, 0.257, 0.200 and 0.1035 ) All outgoing waves must have the same horizontal slowness.

A reflected SV wave will always be possible.

The mode converted P wave in the lower space will have to have $\sin \theta = 3.5$ times (0.306, 0.257, 0.200 and 0.1035 respectively) or $1.07, 0.899, 0.700, 0.362$. The first case leads to an evanescent mode converted P wave in the lower space. All the other incident angles lead to an outgoing P wave in the lower space. The directions of these mode converted lower space P waves are (from the normal) the arcsins, i.e 64, 44, and 21 degrees.

A transmitted SV wave will have to have its $\sin \theta = 4.0$ times (0.306, 0.257, 0.200 and 0.1035 respectively) or $1.22, 1.028, 0.800$ and $0.414$. The first two incident angles lead to evanescence of the transmitted P wave, the latter two lead to propagating transmitted P waves. The arcsines are 53 and 24 degrees.

A transmitted P wave will have to have its $\sin \theta = 7.0$ times (0.306, 0.257, 0.200 and 0.1035 respectively), or $2.14, 1.799, 1.400$ and $0.72$. Only the last case leads to an outgoing transmitted P wave … at an angle form the normal of $\arcsin(0.72) = 46$ degrees.