HW3.1 Consider a string on an elastic foundation with stiffness per length \( \kappa \) and subject to a force \( f(t) \) concentrated at \( x = 0 \)

\[
[\rho \frac{\partial^2}{\partial t^2} - T \frac{\partial^2}{\partial x^2} + \kappa] \psi(x,t) = f(t) \delta(x)
\]

3.1a For \( f = 0 \), what is the dispersion relation, \( \omega \) as a function of (real) \( q \), for a plane wave \( \exp(i\omega t-iqx) \)? What is \( q \) as a function of (real) \( \omega \)? Observe that there is a frequency \( \omega_{\text{cutoff}} \) below which no plane wave propagates; it only evanesces. Sketch a plot of \( \omega \) as a function of \( q \). What is the group velocity \( v_g(\omega) = \frac{d\omega}{dq} \) as a function of \( \omega \)?

3.1b Do a double FT on the above PDE for the case \( f(t) = \delta(t) \) and solve for \( \tilde{G}(q,\omega) \).

Then do the inverse \( q \)-transform by residues (you may have to introduce the \(-i\epsilon \) to \( \omega \)) and determine the frequency domain Greens function for this system \( \tilde{G}(x;\omega) \). Notice that the locations of the poles are qualitatively different depending on whether \( \omega \) is above cutoff or not. You may replace \( x \) with \(|x|\) if you like. You may assume \( \omega \geq 0 \) for simplicity, and use \( \tilde{G}(x; -\omega) = \tilde{G}(x; \omega)^* \) if you need to determine what goes on at negative \( \omega \).

3.1c For the case \( f(t) = \exp(i\omega t) \), at what time average rate is work being done by the force? Show that it is zero if \(|\omega| < \omega_{\text{cutoff}} \). (This makes sense, there is no propagation at such frequencies to carry the energy away.)

3.1d Construct an expression for the response in the time domain \( \psi(x,t) \) as an inverse FT on \( \tilde{G}(x;\omega) \). Show that your integral expression for \( \psi(x,t) \) is real (in spite of initial appearances.) To do this, it is best to convert your integral from \( \omega = -\infty \) to \( \infty \) to an integral from \( \omega = 0 \) to \( \infty \) by recognizing how the integrand changes when \( \omega \) changes sign.

3.1e Evaluate this expression and plot it as a function of \( t \) for a fixed value of \( x = 20 \) (take \( T = \kappa = \rho = 1 \)) by doing the \( \omega \) integration. You may have to do it numerically (thus it is a matter of doing a numerical integration over \( \omega \), and doing so again and again for every time \( t \) of interest. You may find that it is convenient to write out separate expressions for the integrations from 0 to 1 and from 1 to \( \infty \). You may suppress the high frequencies and the problematic convergence of the integral by inserting an ad-hoc factor \( \exp(-\omega^2/\Omega^2) \) that may be interpreted as a low-pass filter; just make sure \( \Omega \gg 1 \) so you don’t lose the interesting physics near and below \( \omega = 1 \). You should be able to observe in your plot of \( \psi(t) \) an apparent relation between the time-of-arrival and the oscillation rate: low frequencies arrive late, in accord with the group velocity derived in 3.1a. You should also be able to observe causality: there should be no signal arriving before the fastest group velocity could carry it \( t_{\text{earliest arrival}} = 20/v_g \text{max.} \)