Elastic Waves  HW 2 due Sept 9, 2015

HW2.1 Given a finite anisotropic body of volume V with a traction free surface S, and a
distribution of body forces in the interior \( f(x,t) \), such that the wave equation is

\[
\rho(\ddot{u}_i) - [c_{ijkl}(x) u_{k,l}]_{,j} = \tilde{f}(x,t)
\]

show that the solution (for any specified initial conditions at time zero) is unique.

HW 2.2 Show that \( \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \) in plane strain (defined as a field of the form
\( \tilde{u} = \hat{u}_x(x,y,t) + \hat{u}_y(x,y,t) \))

HW2.3 Prove the continuity equation for the case of heterogeneous anisotropic medium
with \( c_{ijkl}(x) \)

\[
\frac{\partial}{\partial t} E(\ddot{x};t) + \frac{\partial}{\partial x_i} F_i(\ddot{x};t) = 0
\]

HW2.4 A delta-function pulse \( u_{\text{incident}} = \delta(t-x/c) \) may be Fourier decomposed in the form:

\[
u(x,t) = \frac{1}{2\pi} \int \exp\{i\omega t - i\omega x/c\} d\omega.
\]

where the integral runs from \(-\infty\) to \(\infty\).

The pulse is incident from the left in an infinite string (which has lineal density \( \rho \) per length
and tension T) upon a mass-spring combination as illustrated. For each Fourier component (i.e for
each \( \omega \)) find the transmitted wave at \( x > 0 \)

\( Q(\omega) \exp\{i\omega t + i\omega x/c\} \)

and find the wave at \( x < 0 \) in the form of the sum of incident and the reflected

\( \exp\{i\omega t + i\omega x/c\} + R(\omega) \exp\{i\omega t - i\omega x/c\} \).

Then construct the transmitted wave at \( x > 0 \) \( u_{\text{transmitted}} \) (some function of \( x-ct \)) in the time domain
by doing the Fourier integral re-composition. (It may be evaluated by Cauchy residue theorem.)

You may assume the spring is stiff enough (large enough \( k \)) that the system is underdamped; thus
the two poles of the integrand are located at points \( \omega = \pm \omega_d + i\eta \) with non zero \( \omega_d \) and positive \( \eta \).

The spring \( k \) resists
motion and transmits a force
equal to \( k \) times the
displacement of the mass.
You may wish to note that the
string is continuous at the mass,
but its slope is not. You will
need to construct the "jump"
condition relating the slope at \( x = 0^- \) to the slope at \( x = 0^+ \).

Be careful – your integrand may converge slowly at large \( \omega \). If so, this can be
fixed by adding and subtracting a term of the form \( A \exp(i\omega t)/i\omega \) (whose inverse FT you
know.)