14.1 A one-dimensional random medium.

\[
\rho \tilde{G}(x, x') - \mu G''(x, x') + k(x)G(x, x') = \delta(x - x')\delta(t)
\]

This may be conceptualized as a string with a random stiffness support \( k(x) \)

Take \( k(x) = k_o + \epsilon(x) \) where \( \langle k \rangle = k_o \) is the mean stiffness, and fluctuations \( \epsilon \) have zero mean \( \langle \epsilon \rangle = 0 \). Also take \( \omega_0/\rho = \mu = k_o = 1 \). Take the covariance of the (assumed Gaussian) fluctuations to be \( \langle \epsilon(x)\epsilon(y) \rangle = E \exp(-\beta|x-y|) \) so that \( E = \langle \epsilon^2 \rangle \) and the fluctuations are uncorrelated over distances much larger than \( 1/\beta \).

Use the First-order Smoothing Approximation to construct the operator \( M \) in the spatial domain \( M(x-y; \omega) \), and in the wavenumber domain \( M(q; \omega) \). And thereby obtain the dispersion relation \( f(q, \omega) = 0 \) for the average Greens function.

In order to get insight into the behavior of its root \( q(\omega) \), take \( E \) to be small enough that the dispersion relation is not much modified from that of the bare medium \( G_o \). Also let us assume \( \beta \) is large \( \beta \gg \omega \) (and that \( \omega > 1 \)) this simplifies the results considerably (and corresponds to assuming that the fluctuations in stiffness \( k \) have only short range correlations, mostly confined to distances less than \( 1/\beta \).)

What is the attenuation \( -\text{Im}q(\omega) \)?

Be careful, \( G^o \) is not as it is in the notes; it is affected by \( k_o \).

I start with \( G^o \) (with \( \rho = \mu = 1, k = k_o = 1 \))

\[
\tilde{G}^o(x, \omega) = \frac{1}{2\pi} \int dq \frac{\exp(-iqx)}{-\omega^2 + 1 + q^2}
\]

This may be evaluated by residues, (but replace \( x \) with \( |x| \) because we know it is even in \( x \)) to get

\[
\tilde{G}^o(x, \omega) = \frac{-i}{2q^*(\omega)} \exp(-iq^*(\omega)|x|); \text{ where } q^*(\omega) = \sqrt{\omega^2 - 1}
\]

I also need \( M(x) \), defined as

\[
M(x) = \tilde{G}^o(x, \omega)E \exp(-\beta|x|)
\]

Its spatial FT is

\[
\tilde{M}(q) = \int dx \frac{-i}{2q^*(\omega)} \exp(-iq^*(\omega)|x|)E \exp(-\beta|x|) \exp(iqx)
\]

\[
= \frac{-iE}{2q^*(\omega)} \int \exp(-(\beta + iq^*)|x| + iqx) \ dx = \frac{-iE}{2q^*(\omega)} \frac{2(\beta + iq^*)}{(\beta + iq^*)^2 + q^2}
\]

The integral was done by considering the integration over positive and negative \( x \) separately; each of which ends up being of the form \( \int_0^\infty \exp(-\xi x) \ dx = 1/\xi \)

Our average \( G \) now has the dispersion relation
\[-\omega^2 + 1 + q^2 - \bar{M}(q;\omega) = 0\]

Let us take the solution to be close to \(q^*\): \(q = q^*(\omega) + \delta(\omega)\) where \(\delta\) is of order \(E\), and therefore small.

Then, to leading order in \(\delta\) and \(E\),

\[0 = -\omega^2 + 1 + (q^* + \delta)^2 - \bar{M}(q^* + \delta;\omega) \approx -\omega^2 + 1 + q^{*2} + 2q^*\delta - \bar{M}(q^*;\omega)\]

or,

\[\delta \approx \frac{-iE}{2q^{*2}} \left( \beta + iq^* \right) \]

We see that it has a real and imaginary part; there is attenuation, and a modified wave speed.

At large \(\beta\), it simplifies further:

\[\delta = \frac{-iE}{2\beta q^{*2}}\]

The only effect here is attenuation, i.e., \(q\) gets an imaginary part but its real part is unchanged from that of the bare medium.

14.2 Let us make a quick rough estimate for the rate at which a diffuse field in a reverberant solid loses energy due to radiation into the air. Assume an energy density \(E\) in a solid, of mass density \(\rho\), speeds \(c_L\) and \(c_T\), volume \(V\) and surface area \(A\). \(E_{\text{total}} = EV\). In terms of \(E\), estimate the mean square displacement amplitude in the interior in each direction. Using this value for mean square displacement, but applying it to normal displacements on the surface (there is of course an error here; diffuse waves at surfaces have greater amplitude than in interiors) and assuming the motion on the surface is piston-like – varying only weakly in tangential directions (this is not too bad an approximation, wave speeds and therefore wavelengths in the solid are much greater than in the air), calculate the rate at which power is radiated into the air. You will need to perform a side calculation of the intensity output of a wide piston in air; use the wave equation in the air with a speed \(c_{air}\) and mass density \(\rho_{air}\).

For \(V = 1000\) cm\(^3\), \(A = 600\) cm\(^2\), in aluminum with \(c_L = 6000\) m/s and \(c_T = 3000\) m/s, and \(\rho_{aluminum} = 2700\) kg/m\(^3\) and \(c_{air} = 343\) m/sec and mass density \(\rho_{air} = 1.3\) kg/m\(^3\), find the time scale \(\tau\) over which diffuse energy decays by a factor of \(e\) due to this mechanism. It should be independent of frequency, and scale with the surface area \(A\), and the ratio of mass densities and scale inversely with the volume \(V\). Assuming a transit time \(L/c_T\) (with \(L = 10\) cm) how many transits does a ray achieve in time \(\tau\)? (This is a measure of the reverberance.)

We take the mean square displacement in any given direction, in the interior to be, as in the notes, \((E/3\rho\omega^2V)\) and take this to also describe mean square normal motions at the surface.

A side calculation of a pressure wave in the air propagating in the \(z\) direction away from the solid surface has \(p = P_o \exp(\omega t - \omega x/c_{air})\) and displacement \(u\) (in the air in the \(z\)-direction) equal to \(U_o \exp(\omega t - \omega x/c_{air})\). Force balance in the air (\(\rho_{air} \frac{\partial^2 u}{\partial t^2} \sim \text{Del} p\)) says that \(\omega^2 P_o/c_{air} = \rho_{air} U_o \omega^2\). So the amplitude of the pressure and displacement are related: \(P_o = -i\rho_{air} \omega U_o c_{air}\). The area density of power (its intensity) applied from the solid onto the air is the time average of pressure times normal velocity. But pressure at the surface is simply \(\rho c_{air}^2\) times normal velocity (\(\omega U_o\)), so time average Intensity is...
\[ \rho_{\text{air}} c_{\text{air}} \omega^2 \langle u^2 \rangle = (\rho_{\text{air}} c_{\text{air}} / 3\rho) (E/V) \]. Multiply by surface area \( A \) to get total power into the air

\[ \Pi = (1/3) c_{\text{air}} (\rho_{\text{air}} / \rho) (A/V) E \]  
Thus the energy \( E \) diminishes like \( dE/dt = -\Pi \), or \( E \sim E_0 \exp(-t/\tau) \) with \( 1/\tau = (1/3) c_{\text{air}} (\rho_{\text{air}} / \rho) (A/V) \).

For a cube, \( A/V = 6/L \), \( \rho = 2700; \) pair = 1.3, \( L = 0.1 \) meter, we find \( \tau = 0.3 \) seconds. \( c_{\text{air}} = 343; \) This is much longer than we usually observe diffuse fields over. Other mechanisms dissipate them faster.

Let us compare this with a transit time over 0.1 meters at a shear wave speed: 33 \( \mu \) sec. The

\textit{reverberance} (assuming no other dissipation mechanisms) is the ratio of these times and would be about \( 10^4 \).

14.3 What fraction of the diffuse wave energy near frequency \( \omega \) in a thin plate (of thickness \( H << \omega/c_T \)) is in the form of longitudinal-in-plane 'plate' waves? The plate supports bending waves with dispersion relation \( \omega = \alpha q^2 H c_T \), \( \text{SH0 waves} \) with dispersion relation \( \omega = q c_T \), and longitudinal plate waves with dispersion relation \( \omega = \beta q c_T \). \( \alpha, \beta \) are dimensionless quantities of order unity that depend on Poisson ratio.

Three wave types, with dispersion relations

\[ q = \omega / c_T \] \( \text{SH0 waves} \)
\[ q = \omega / \beta c_T \] \( \text{'plate' waves ( also called S0 Lamb waves)} \)
\[ q = [ \omega / \alpha H c_T ]^{1/2} \] \( \text{bending waves, or flexural waves (also called A0 Lamb waves)} \)

Each wave type contributes to the number of modes an amount \( N = \pi [q(\omega)L/2\pi]^2 \).

So \( N_{\text{total}} \) is

\[ N_{\text{total}} = [A/4\pi] (q_{\text{SH0}}^2 + q_{\text{S0}}^2 + q_{\text{A0}}^2) \]

If we confine ourselves to a frequency band at \( \omega \) with width \( \Delta \omega \), we get a number of modes

\[ N_{\text{total}}(\omega) \Delta \omega = \Delta \omega [A/4\pi] (2\omega/c_T^2 + 2\omega/\beta^2 c_T^2 + 1/\alpha H c_T) \]

The fraction in plate waves is the ratio of the second of these terms to the sum of them. At low frequencies, the ratio is small….most energy is in bending waves. loosely speaking we would say this is because they have shorter wavelength so you can fit more of them into the same region.

14.4 Two solid bodies of equal volume and equal Poisson ratio are in contact. \( \text{Solid#1} \) has one quarter the mass density but twice the wave speed of \( \text{solid#2} \) (so they have equal moduli.) Which body has more diffuse energy density? In which body is the greater mean square displacement at typical interior points?

The number of modes scales in 3-d with the volume and the frequency^2 and the inverse cube of the speeds, so solid #2 has \textit{eight} times the energy per volume. It has four times the mass density, so its mean square displacement (equal to \( E/3\rho V \)) is only twice as great.