Elastic Waves  HW 13  solution

13.1  Starting with the equations of isotropic linear thermo-elasticity
\[
\sigma_{ij} = \lambda u_{i,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i}) - \eta (T - T_{ref}) \delta_{ij}
\]
\[
\rho C_v \dot{T} = \kappa V^2 T + Q - \eta T_{ref} \dot{u}_{k,k}
\]
\[
\partial_i \sigma_{ij} = \rho \ddot{u}_j
\]
where \(\lambda, \mu\) are the isothermal moduli.

Find the speed of longitudinal adiabatic waves (i.e., the speed when \(\kappa\) goes to zero.) What is the ratio of displacement amplitude to temperature amplitude for such waves?

(are the temperature and displacement in phase or out of phase with each other?)

Seek longitudinal plane wave solutions, and, as on p 197, get
\[
\begin{bmatrix}
k^2(\lambda + 2\mu) - \rho \omega^2 - i k \eta \\
\eta T_{ref} \omega k \rho C_v i \omega + \kappa k^2
\end{bmatrix}
\begin{bmatrix}
U \\
\Theta
\end{bmatrix} = 0
\]

It was not asked for, but at \(\kappa = \infty\) (infinite thermal conductivity) we must have \(\Theta = 0\) (i.e., no temperature variations; they all get conducted away) This implies \(k^2(\lambda + 2\mu) - \rho \omega^2 = 0\) … which gives the speed of longitudinal waves in terms of the isothermal moduli.

If \(\kappa = 0\) on the other hand, i.e., no thermal conductivity, then the bottom line tells us \(\eta T_{ref} \omega k U = -\rho C_v i \omega \Theta\) (which implies that the temperature and displacement oscillations are 90 degrees out of phase. It also tells us that the temperature oscillations have amplitude \(|\Theta|\) equal to \(\eta T_{ref} k / \rho C_v\) times the displacement amplitude \(|U|\).)

The upper equation tells us
\[
\{k^2(\lambda + 2\mu) - \rho \omega^2\} U - i k \eta \Theta = 0
\]
therefore, because \(\eta T_{ref} \omega k U = -\rho C_v i \omega \Theta\) from the lower equation
\[
\{k^2(\lambda + 2\mu) - \rho \omega^2 - i k \eta (\eta T_{ref} \omega k / -\rho C_v i \omega)\} U = 0
\]
i.e.,
\[
k^2(\lambda + 2\mu) - \rho \omega^2 + k^2 \eta^2 (T_{ref} / \rho C_v) = 0
\]

We see the dispersion relation we get at \(\kappa = 0\). The effective longitudinal modulus \(\lambda + 2\mu\), and speed wave, is greater than it was at infinite \(\kappa\).
13.2 Homogeneous medium thermo-elastodynamics has a material-dependent characteristic time scale
\[ \tau = \frac{D}{c_L^2}. \]
Where D is heat diffusivity (units of m^2/sec) \( D = \frac{\kappa}{\rho C_V}. \)

What is that time scale for aluminum? (You'll need to look up the properties of aluminum. Be careful about units/check your dimensions. Thermal properties are reported in many different guises.) (Recall \( \eta \) is a simply related to the thermal expansion coefficient.)

I use \( \kappa = 205 \) Watt/m°K, \( \rho = 2700 \)kg/m^3 and \( C_V = 900 \)J/kg°K to get \( D = 8.4 \times 10^{-5} \) m^2/sec. Using \( c_L = 6000 \)m/sec, I find \( \tau = 2.3 \times 10^{-12} \) sec.

For low frequencies \( \omega \ll 1/\tau \), show that the longitudinal wave (the one with speed\(^2\) close to \((\lambda + 2\mu)/\rho \)) has almost the adiabatic speed.

We again invoke,

\[
\begin{bmatrix}
k^2(\lambda + 2\mu) - \rho \omega^2 & -i k \eta \\
\eta T_{ref} \omega k & \rho C_V i \omega + \kappa k^2
\end{bmatrix}
\begin{bmatrix}
U \\
\Theta
\end{bmatrix} = 0
\]

or

\[
\det\begin{bmatrix}
k^2(\lambda + 2\mu) - \rho \omega^2 & -i k \eta \\
\eta T_{ref} \omega k / \rho C_V & i \omega + D k^2
\end{bmatrix} = 0
\]

The det is a quadratic in \( k^2 \), so we can get closed form solutions for \( k \) as a function of \( \omega \) for any material parameters. We can avoid having to deal with the quadratic equation by the following argument: Whatever the details, our wave will have \( k \sim \omega/c \), so the term \( i \omega + D k^2 \approx i \omega [1 - i \omega \tau] \approx i \omega \) (when \( \omega \tau \ll 1 \)) so this low frequency limit is the same as the adiabatic limit.

Is this the relevant regime for MHz ultrasonics?

If \( \omega \ll 1/\tau \) then \( f \ll 68 \) GHz. So this IS the relevant regime for most ultrasonics)

In this regime \( \omega \ll 1/\tau \), what is the dimensionless attenuation/wavenumber = -Im k/Re k as a function of \( \omega \)? Evaluate this in aluminum at \( f = 1 \) MHz, \( \omega = 2\pi \) f.

As argued above, in this limit, we can rewrite our dispersion relation as (valid as long as \( k \sim \omega/c_{adiabatic} \))
\[
\begin{align*}
\det\left[k^2 c_{\text{isothermal}}^2 - \omega^2 - k\eta / \rho \right] + \eta T_{ef} k / \rho C_V \left[1 + i\omega \tau \right] = 0 \\
\text{or,} \\
k^2 c_{\text{adiabatic}}^2 - \omega^2 + i\omega \tau (k^2 c_{\text{isothermal}}^2 - \omega^2) = 0
\end{align*}
\]

This can be solved for \( \text{Im} \, k \). The effect is weak, because the imaginary term is small, not only because of the factor of \( i\omega \tau \), but also because of the other factor. I get \( \text{Im} k / k \) at MHz to be about \( 10^{-8} \).

13.3 Given a Maxwell–model material and using units such that speed is unity at high frequency, and intrinsic time scale \( \tau = 1 \), plot the real part of the speed, and the attenuation (equal to \( -\text{Im}(\omega / \text{speed}) \)) versus frequency.

In the Maxwell model medium, complex wave speed \( \omega / k \) is \( [1+1/i\omega]^{-1/2} \). (p193) At low \( \omega \) this is \( c = (1+i) \left[ \omega / 2 \right]^{-1/2} \). At large \( \omega \) it is \( c = 1 - 1/2i\omega \). A plot of the real part of \( c \) is like a \( \sqrt{\omega} \) at low \( \omega \) asymptoting to a constant 1 at high \( \omega \). A plot of \(-\text{Im}(\omega / c)\) is \( \sqrt{\omega} / 2 \) at low \( \omega \) (i.e. like a \( \sqrt{\omega} \)) and asymptoting to a constant at large \( \omega \).

13.4 It is desired to construct a filter \( f(t) \) that attenuates a signal by 90% over a frequency band from \(-1\) to \(+1\), and to leave it unchanged for all other frequencies (a low-pass filter). The magnitude of its FT is plotted here.

Assuming that \( f(\omega) \) has no zeros in the LHP, (so that its log is analytic there) find the required phase function \( \phi(\omega) \) such that \( f=F\exp(i\phi(\omega)) \) is the FT of a function \( f(t) \) that is causal.

\[
\text{Re} \log f = \log F = \ln(1) \quad \text{for} \quad |\omega|<1, \quad = 0 \quad \text{otherwise.}
\]

The real and imaginary parts should be related:

\[
\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\text{Re} \log f(\omega)d\omega}{\omega - z} = \frac{1}{\pi} P \int_{-1}^{1} \frac{(\ln 1)d\omega}{\omega - z} = \text{Im} \log f(z) = \phi(z)
\]

so

\[
\phi(z) = \frac{1}{\pi} P \int_{-1}^{1} \frac{(\ln 1)d\omega}{\omega - z} + \frac{1}{\pi} P \int_{z+\epsilon}^{1} \frac{(\ln 1)d\omega}{\omega - z} = -2.3 \left[ \int_{-1}^{z-\epsilon} \frac{d\omega}{\omega - z} + \int_{z+\epsilon}^{1} \frac{d\omega}{\omega - z} \right]
\]

\[
= -2.3 \ln \left| \frac{1-z}{1+z} \right|
\]