12.1 A plane P wave is incident in the xy plane upon a rigid and infinitely dense circular cylinder of infinite length in the z-direction, and radius \( a \). (Think of it as an approximation to a steel rod in plastic.) The matrix medium is isotropic with density \( \rho \), and moduli \( \lambda, \mu \). (Real part suppressed)

\[
\Phi^{\text{incident}} = \exp(-i\omega x / c_L) \exp(i\omega t) = \exp(-i\omega \cos \theta / c_L) \exp(i\omega t)
\]

12.1a) Find the incident intensity (energy per time per y-z area) of this wave. The easy way to do this is to evaluate the time-average Kinetic energy density \( \frac{1}{4} \rho |\nabla \Phi|^2 \) (using \( ru = \text{Re} r \nabla \Phi \)) doubling it to account for strain energy density, and then multiplying by speed.

The incident wave's material velocity field is

\[
\tilde{u} = \text{Re} \tilde{\nabla} \Phi \]

Now multiply by \( (\rho/2) \) to get kinetic energy density, then by 2 to get total energy density, and then by speed to get intensity (energy per time per area) \( I = \rho \omega^4 / 2c_L \). We conclude that a plane \( \Phi \)-wave carries an intensity \( \rho \omega^4 / 2c_L \) times the absolute value square of its amplitude. The calculation for a plane S wave of unit amplitude \( \tilde{H} = \hat{k} \exp(-i\omega x / c_T) \exp(i\omega t) \) would be similar, giving an intensity \( \rho \omega^4 / 2c_T \).

Write the incident Helmholtz potential \( \Phi \) as a partial wave sum (now suppressing the \( \exp(i\omega t) \) )

\[
\Phi^{\text{incident}}(\vec{r}) = a_0 J_0 (\omega r / c_L) + 2 \sum_{l>0} a_l J_l (\omega r / c_L) \cos(l\theta) = \sum_{l=-\infty}^{\infty} a_l J_l (\omega r / c_L) \exp(il\theta)
\]

with \( a_l = a_{-l} \).

12.1b) Refer to the course notes and find the complex coefficients \( a_l \).

(see page 180, \( a_l = i^{-|l|} \))

The scattered fields \( \Phi \) and \( \tilde{H} = k \tilde{H} \) (the solenoidal Helmholtz potential, not the Hankel function), such that

\[
\tilde{u} = \text{Re} (\nabla \Phi(r, \theta) + \nabla \times \hat{k} \tilde{H}(r, \theta))
\]

must be of the form of outgoing waves:

\[
\Phi^{\text{scattered}}(\vec{r}) = \sum_{l=-\infty}^{\infty} c_l H_l^{(2)}(\omega r / c_L) \exp(il\theta); \quad \tilde{H}^{\text{scattered}}(\vec{r}) = \hat{k} \sum_{l=-\infty}^{\infty} d_l H_l^{(2)}(\omega r / c_T) \exp(il\theta)
\]
Symmetry tells us* that \( c_i = c_{-i} \) and \( d_i = -d_{-i} \) .... but these conditions should follow from the scattering and need not be imposed now. (* because \( H_l \) must be odd in \( \theta \) so that the associated \( u_r \) will be even in \( \theta \), therefore \( d_i = -d_{-i} \))

12.1c) Invoke the rigid boundary conditions at \( r = a \) to solve for the outgoing coefficients \( c \) and \( d \).

The displacement associated with the \( \Phi \) waves is \( \ddot{u}^{\text{prop}} = \ddot{\Phi} = \hat{r} \partial_\phi \Phi + \hat{\theta} \frac{1}{r} \partial_\theta \Phi \) is

\[
\ddot{u}^{\text{prop}} = \hat{r} \sum_{l=-\infty}^{\infty} k_l [a_l J_i(k_l r) + c_l H_i^{(2)}(k_l r)] \exp(il\theta) + \hat{\theta} \sum_{l=-\infty}^{\infty} (il/r)[a_l J_i(k_l r) + c_l H_i^{(2)}(k_l r)] \exp(il\theta)
\]

The displacement associated with the \( S \) wave is \( \ddot{u}^{\text{Sprop}} = \vec{\nabla} \times \hat{k} H = (\hat{\imath}/r) \partial_\phi \Phi - \hat{\theta} \partial_\theta \Phi \)

\[
\ddot{u}^{\text{Sprop}} = \hat{r} \sum_{l=-\infty}^{\infty} (il/r)d_l H_i^{(2)}(k_l r) \exp(il\theta) - \hat{\theta} \sum_{l=-\infty}^{\infty} k_l d_l H_i^{(2)}(k_l r) \exp(il\theta)
\]

The total displacement at \( r = a \) must vanish, so

\[
u_r(r = a) = \sum_{l=-\infty}^{\infty} e^{il\theta} k_l [a_l J_i(k_l a) + c_l H_i^{(2)}(k_l a)] + e^{il\theta} (il/a)d_l H_i^{(2)}(k_l a) = 0
\]

\[
u_\theta(r = a) = \sum_{l=-\infty}^{\infty} e^{il\theta} (il/a)[a_l J_i(k_l a) + c_l H_i^{(2)}(k_l a)] - e^{il\theta} k_l d_l H_i^{(2)}(k_l a) = 0
\]

Each term \( \exp(il\theta) \) must vanish separately (by the linear independence of the \( \exp(il\theta) \)). Thus, for every \( l \),

\[
k_l c_l H_i^{(2)}(k_l a) + (il/a)d_l H_i^{(2)}(k_l a) = -k_l a_l J_i(k_l a)
\]

\[
(il/a)c_l H_i^{(2)}(k_l a) - k_l d_l H_i^{(2)}(k_l a) = -(il/a)a_l J_i(k_l a)
\]

This can be solved for the \( c_i \) and \( d_i \),

\[
c_i = a_i \frac{k_l J_i(k_l a)k_l H_i^{(2)}(k_l a) - (il/a)^2 H_i^{(2)}(k_l a) J_l(k_l a)}{-k_l H_i^{(2)}(k_l a)k_l H_i^{(2)}(k_l a) + (il/a)^2 H_i^{(2)}(k_l a) H_i^{(2)}(k_l a)}
\]

\[
d_i = a_i (il/a) k_l \frac{-J_l(k_l a) H_i^{(2)}(k_l a) + H_i^{(2)}(k_l a) J_l(k_l a)}{-k_l H_i^{(2)}(k_l a)k_l H_i^{(2)}(k_l a) + (il/a)^2 H_i^{(2)}(k_l a) H_i^{(2)}(k_l a)}
\]

It is apparent that the \( d_l \) are odd in \( l \), as anticipated; in particular \( d_0 = 0 \). The \( c_i \) are even in \( l \).

12.1d) Find the total power (per unit length in the \( z \) direction) in the scattered wave field (you may do this by integrating scattered field outgoing intensity over a big circle at large \( r = R \), using asymptotic expressions for the Hankel functions) and construct an expression for the ratio of this to the incoming intensity. This is the "cross section" with units of length. Be careful: the outgoing power is a sum over the \( |c_i|^2 \) and the \( |d_l|^2 \) but the \( c \) and \( d \) may enter with different pre-factors. What fraction of the cross section is due to mode-conversion into outgoing SV waves?
Each scattered partial wave looks, at large distances \( r \), like an outgoing plane wave. It carries intensity (as discussed above) equal to its \( |\text{complex amplitude}|^2 \) times \( \rho \omega^4 / 2c_{\text{LorT}} \). Its complex amplitude is \( H_\ell(kr) \), asymptotically equal to \( [2/\pi kr]^{1/2} \). Thus the total scattered power is (after multiplying by circumference \( 2\pi r \) and realizing the different partial waves do not interfere with each other because of orthogonality)

\[
\Pi^{\text{scattered}} = 2\pi r \sum |c_\ell|^2 \frac{2}{\pi k_Lr} (\rho \omega^4 / 2c_L) + 2\pi r \sum |d_\ell|^2 \frac{2}{\pi k_T r} (\rho \omega^4 / 2c_T)
\]

\[
= 2\rho \omega^3 \sum |c_\ell|^2 + |d_\ell|^2
\]

It appears the S and P waves contribute with the same prefactors. (I did not expect that. On p186 of the notes, the prefactor \( \pi \) was found to be proportional to \( vk \). But if \( v = c \) as it does here, this becomes \( \omega \) and both prefactors are identical, so maybe I shouldn’t have been surprised.) Divide by incident intensity \( I = \rho \omega^4 / 2c_L \) to get cross section. Clearly the fraction of the scattered power due to shear waves is

\[
\frac{\sum |d_\ell|^2}{\sum |c_\ell|^2 + |d_\ell|^2}.
\]

12.1e) In the limit of a small cylinder \( ka = z << 1 \), use \( J_0(z) = 1 \) and \( H_\ell^{(2)}(z) = 1-i(2/\pi) \[ \gamma + \log(z/2) \] \) (\( \gamma \) is Euler's constant). Find the cross section.

Assuming the \( l=0 \) term is the only important one\(^*\) in this limit, we find that the \( d_0 = 0 \), and \( c_0 \) is about (using \( J = 1, J' = -z, H = -2i/\pi \log z ; H' = -2i/\pi z \)) \( c_0 = i\pi(k_L a)^2 \). The scattered power is therefore \( \Pi^{\text{scattered}} = 2\rho \omega^3 i\pi (k_L a)^2 \) and the cross section is \( \Pi^{\text{scattered}} / \rho \omega^4 / 2c_L \) is proportional to \( a^4 \), and much less than the geometric cross section 2a.

\(^*\) actually the \( c_1 \) and \( d_1 \) terms are of the same order, so they oughtn't have been ignored.

12.1f) Confirm that your results satisfy energy conservation. To do this you may wish to introduce (and calculate) quantities I call \( \pi^P_m \) and \( \pi^S_m \) — that describe the outgoing (or incoming) power in the \( m^{th} \) partial wave \( H^{(1 \text{ or } 2)}_m \cos m\theta \) of each type P and S. Then show that energy conservation demands

\[
\Pi^{\text{incoming}} = \sum |a_\ell| / 2 \ell^2 \pi^P_\ell = (\Pi^{\text{outgoing}} = \sum |a_\ell| / 2 + c_\ell \ell^2 \pi^S_\ell + \sum |d_\ell|^2 \pi^S_\ell
\]

Not sure how easy it is to confirm energy conservation, you may need some Bessel function identities.

The total field may be written as
\[ \Phi^{\text{total}}(\vec{r}) = \sum_{l=-\infty}^{\infty} [(c_l + a_l / 2) H_l^{(2)}(\omega r / c_L) + (a_l / 2) H_l^{(1)}(\omega r / c_L)] \exp(il\theta) \]

\[ \tilde{H}^{\text{total}}(\vec{r}) = \hat{k} \sum_{l=-\infty}^{\infty} d_l H_l^{(2)}(\omega r / c_L) \exp(il\theta) \]

We see here that the incoming part is \((a_l / 2) H_l^{(1)}(\omega r / c_L)\). The outgoing part is the other terms. As each wave carries power \(2\rho\omega^3\) times its coefficient \(\lambda^2\), we conclude

\[ \Pi^{\text{incoming}} = \sum |a_l| / 2^l \pi^p_l = \sum |a_l| / 2 + c_l |l^2 \pi^p_l + \sum |d_l| |l^2 \pi^s_l \]

with \(\pi^p_l = \pi^s_l = 2\rho\omega^3\). Proving that the above coefficients \(c, d\) satisfy this looks to be very hard, especially for arbitrary \(a\). Probably easier just to show it numerically.

12.2 Use the first Born approximation to solve for the far-field scattered wave from a plane scalar wave in a 3-d fluid incident upon a small inclusion in an otherwise homogeneous fluid.

\[
\Phi(\vec{r}) - \kappa(\vec{r}) \nabla \cdot \left[ \frac{1}{\rho(\vec{r})} \nabla p(\vec{r},t) \right] = -s(\vec{r},t)
\]

where \(s\) is a source distribution, \(p\) is acoustic pressure, \(\kappa\) is bulk modulus and \(\rho\) is fluid density. Let us take the density \(\rho=\text{constant} = 1\). Let us go to the frequency domain and let us take \(\kappa\) in the form \(\kappa^{-1} = 1 - \epsilon(\vec{r})/\omega^2\) where \(\epsilon\) has support only in the inclusion. Then the PDE becomes

\[ \nabla^2 p(\vec{r}) + \omega^2 p(\vec{r}) = s(\vec{r}) + \epsilon(\vec{r}) p(\vec{r}) \]

\(s(\vec{r})\) is the (far to the left) source of the field. Take the inclusion to be a small irregular region of nominal radius \(a \ll 1/\omega\) and volume \(V\). You will need the Greens function for the bare medium that satisfies \(\nabla^2 \hat{G}(\vec{r},\vec{r}') - \omega^2 \hat{G}(\vec{r},\vec{r}') = \delta^3(\vec{r} - \vec{r}')\) and is equal to \(\hat{G}(\vec{r},\vec{r}';\omega) = \exp(-i\omega r)/4\pi r\) where \(r = |\vec{r} - \vec{r}'|\)

At the first Born approximation, show that the scattered wave is, for small volume, approximately spherical, with no significant angular dependence.

First Born has it that \(p^{\text{scattered}}(\vec{r}) = \int G^0(\vec{r},\vec{r}') \epsilon(\vec{r}') \ p^{\text{incident}}(\vec{r}') \ d^3 r'\). In doing the integral over \(r'\) (and for observation points \(r\) far from the scattering region) and because the region in which \(\epsilon \neq 0\) is small, the factors of \(p^{\text{incident}}\) and \(G^0\) may be approximated by their values at the center of the region (we'll put our origin there). Then
The integral here may be defined as E, a measure of the total amount of "epsilonness," equal to average ε times the volume of the region. 

E\(=\langle\epsilon\rangle V\). Then \(p^{\text{scattered}}(\vec{r}) = -\frac{\exp(-ior)}{4\pi r} E\) This is manifestly independent of direction, so it is a spherically symmetric field.

What is the total cross section?

The outgoing intensity is \(|p^{\text{scattered}}(\vec{r})|^2\) (times a prefactor that we may ignore) i.e,

\[ I^{\text{scattered}} = \frac{E^2}{16\pi^2 r^2} \]

We integrate this over all directions (i.e multiply by \(4\pi r^2\)) to get scattered power \(\Pi^{\text{scattered}} = E^2 / 4\pi\). Then we divide by incident intensity \(|p^{\text{incident}}(r)|^2\) (again ignoring the prefactor) to get cross section \(\sigma = E^2 / 4\pi\). In the small volume limit, we see that this scales with the 4th power of frequency and with the mean square fluctuation in inverse \(\kappa\).