12.1 A plane P wave is incident in the xy plane upon a rigid and infinitely dense circular cylinder of infinite length in the z-direction, and radius \( a \). (Think of it as an approximation to a steel rod in plastic.) The matrix medium is isotropic with density \( \rho \), and moduli \( \mu, \lambda \). (Real part suppressed)

\[
\Phi^{\text{incident}} = \exp(-i\omega x)\exp(i\omega t) = \exp(-i\omega r \cos \theta)\exp(i\omega t)
\]

12.1a) Find the incident intensity (energy per time per y-z area) of this wave. The easy way to do this is to evaluate the time-average kinetic energy density

\[
\frac{1}{2} \rho |\nabla \Phi|^2
\]

(\text{using } \vec{u} = \text{Re } \nabla \Phi)

doubling it to account for strain energy density, and then multiplying by speed.

Write the incident Helmholtz potential \( \Phi \) as a partial wave sum (now suppressing the \( \exp(i\omega t) \))

\[
\Phi^{\text{incident}}(\vec{r}) = a_0 J_0(\omega r / c_L) + 2 \sum_{l>0} a_l J_l(\omega r / c_L) \cos(l\theta) = \sum_{l=-\infty}^{\infty} a_{il} J_l(\omega r / c_L) \exp(il\theta)
\]

(We do not need sine \( \theta \) terms, because the field is symmetric in y)

12.1b) Refer to the course notes and find the complex coefficients \( a_l \).

The scattered fields \( \Phi \) and \( \vec{H} = k \vec{H} \) (the solenoidal Helmholtz potential, not the Hankel function), such that \( \vec{u} = \text{Re } (\nabla \Phi(r, \theta) + \nabla \times \hat{k} \vec{H}(r, \theta)) \) must be of the form of outgoing waves:

\[
\Phi^{\text{scattered}}(\vec{r}) = \sum_{l=-\infty}^{\infty} c_{il} H_l^{(2)}(\omega r / c_L) \exp(il\theta); \quad \vec{H}^{\text{scattered}}(\vec{r}) = \hat{k} \sum_{l=-\infty}^{\infty} d_{il} H_l^{(2)}(\omega r / c_T) \exp(il\theta)
\]

(We still do not need sine terms, because the scatterer and the incident field are symmetric in y)

12.1c) Invoke the rigid boundary conditions at \( r = a \) to solve for the outgoing coefficients \( c \) and \( d \).

12.1d) Find the total power (per unit length in the z direction) in the outgoing wave field (you may do this by integrating scattered field outgoing intensity over a big circle at large \( r = R \), using asymptotic expressions for the Hankel functions) and construct an expression for the ratio of this to the incoming intensity. This is the "cross section" with units of length. Be careful: the outgoing power is a sum over the \( |c|_l^2 \) and the \( |d|_l^2 \) but the \( c \) and \( d \) enter with different pre-factors. What fraction of the cross section is due to mode-conversion into outgoing SV waves?

\[\text{see over}\]
12.1e) In the limit of a small cylinder $ka = z \ll 1$, use $J_0(z) = 1$ and $H_0^{(2)}(z) = 1 - i(2/\pi)\left[\gamma + \log(z/2)\right]$ ( $\gamma$ is Euler's constant ) Find the cross section.

12.1f) Confirm that your results satisfy energy conservation. To do this you may wish to introduce (and calculate) quantities I call $\pi^P_m$ and $\pi^S_m$ – that describe the outgoing (or incoming) power in the $m^{th}$ partial wave $H_{1\text{ or }2}^m \cos \theta$ of each type P and S. Then show that energy conservation demands

$$\Pi^{\text{incoming}} = \sum |a_i|/2 \pi^P_i = (7) \Pi^{\text{outgoing}} = \sum |a_i|/2 + |c_i|^2 \pi^P_i + \sum |d_i|^2 \pi^S_i$$

Not sure how easy it is to confirm energy conservation, you may need some Bessel function identities.

12.2 ) Use the first Born approximation to solve for the far-field scattered wave from a plane scalar wave in a 3-d fluid incident upon a small inclusion in an otherwise homogeneous fluid.

The acoustic PDE for an inhomogeneous medium is writable as

$$\ddot{p}(\vec{r}, \omega) - \kappa(\vec{r}) \nabla \cdot \left[ \frac{1}{\rho(\vec{r})} \nabla p(\vec{r}, \omega) \right] = -s(\vec{r}, \omega)$$

where $s$ is a source distribution, $p$ is acoustic pressure, $\kappa$ is bulk modulus and $\rho$ is fluid density. Let us take the density $\rho = \text{constant} = 1$. Let us go to the frequency domain and let us take $\kappa$ in the form $\kappa^{-1} = 1 - \epsilon(\vec{r})/\omega^2$ where $\epsilon$ has support only in the inclusion. Then the PDE becomes

$$\nabla^2 p(\vec{r}) + \omega^2 p(\vec{r}) = s(\vec{r}) + \epsilon(\vec{r}) p(\vec{r})$$

$s(\vec{r})$ is the (far to the left) source of the field. Take the inclusion to be a small irregular region of nominal radius $a \ll 1/\omega$ and volume $V$. You will need the Greens function for the bare medium that satisfies $\nabla^2 G(\vec{r}, \omega) - \omega^2 G(\vec{r}, \omega) = \delta^3(\vec{r} - \vec{r}')$ and is equal to $G(\vec{r}, \vec{r}'; \omega) = -\exp(-i\vec{r}\cdot\vec{r}')/4\pi r$ where $r = |\vec{r} - \vec{r}'|$

At the first Born approximation, show that the scattered wave is, for small volume, approximately spherical, with no significant angular dependence.

What is the total cross section?