Elastic Waves  HW 11  Due Wednesday Nov 11, 2015

An impulsive anti-plane shear line load (of unit impulse-per-length in the z direction) acts in the interior (at height y = a) of a plate of thickness H. (you may wish to refer to the notes' discussion of SH waves in a plate p 137 ff, however there my x-axis was at the midplane, not on the bottom so you'll need to make some changes)

We seek the displacement response \( \tilde{k} q(x,y,t) \) that satisfies the PDE:

\[
\rho \ddot{q}(x,y,t) - \mu \nabla^2 q(x,y,t) = \delta(x)\delta(y-a)\delta(t)
\]

and free BCs on the top and bottom:

\[
\partial_y q(x,y,t)_{y=H} = \partial_y q(x,y,t)_{y=0} = 0;
\]

and rigid BCs at (large) \( x = \pm L \):

\[
q(x,y,t)_{x=\pm L} = 0;
\]

11.1) Find a complete set of (real) modes \( u^{(n,b)}(x,y) \) and natural frequencies that satisfy the BCs and

\[
-\rho \omega^2 u^{(n,b)}(x,y) - \mu \nabla^2 u(x,y) = 0
\]

(for this problem it suffices to find the modes that are even in x, why?) The modes and the natural frequencies will have two indices (here I call them n, b; your mileage may vary), one denoting their horizontal wavenumber \( \xi \), another denoting their branch b.

11.2) Express the response \( q(x,y,t) \) as a sum of modes

\[
q(x,y,t) = \sum_{\text{modes } b,n} \eta_{bn}(t) u^{(bn)}(x,y)
\]

By substituting into the PDE \( \rho \ddot{q}(x,y,t) - \mu \nabla^2 q(x,y,t) = \delta(x)\delta(y-a)\delta(t) \) and knowing

\[
-\rho \omega^2 u^{(bn)}(x,y) - \mu \nabla^2 u(x,y) = 0
\]

and invoking orthogonality, find an ODE for the \( \eta \) and solve it.

11.3) Using your result for \( \eta(t) \) in \( q(x,y,t) = \sum_{\text{modes } b,n} \eta_{bn}(t) u^{(bn)}(x,y) \), and taking L to infinity, write q in the form (in which dependence on L drops out)

\[
q(x,y,t) = \sum_{\text{branches } b} \int_0^\infty d\xi \ f_b(x,y,\xi) \sin(\omega_b(\xi)t)
\]

Check the dimensions, does it indeed have units of displacement per (impulse-per-length) as it ought?

11.4) Evaluate this integral numerically for the response at \( x = 5H, y = 0 \). Take \( \rho = \mu = H = 1, a=0 \), and sum over the lowest 5 branches only and smoothly truncate your integral in \( \xi \) by inserting a factor of \( \exp(-\xi^2H^2/100) \) Find q(t) at this position for times t between 0 and 10. Your result should be zero for times before 5.

11.5) Take the sine-FT (i.e the -Im of the response in the frequency domain) of

\[
q(x,y,t) = \sum_{\text{branches } b} \int_0^\infty d\xi \ f_b(x,y,\xi) \sin(\omega_b(\xi)t)
\]

(Use \( \int_0^\infty \sin(at)\sin(bt)dt = (\pi / 2)[\delta(a-b) - \delta(a+b)] \))

and write it as a sum over branches that propagate at the frequency of interest. For \( H=\rho=\mu=1, y = a = 0, x = 0 \) (i.e. at the position of the source and for the source at the bottom) plot this versus \( \omega \) for \( \omega \) up to 10. (This is of course related to the frequency dependent rate of work done by a harmonic load, \( \Pi \)).