Topological origin of equatorial waves

Delplace, Pierre, J. B. Marson, and Antoine Venaille. (2017). *Topological origin of equatorial waves.* Science: 1075-1077.

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Outline

- **Background**
- Summary
- Critical evaluation
- Citations and future work
- Conclusions

Classical methods are sufficient to characterize equatorial waves

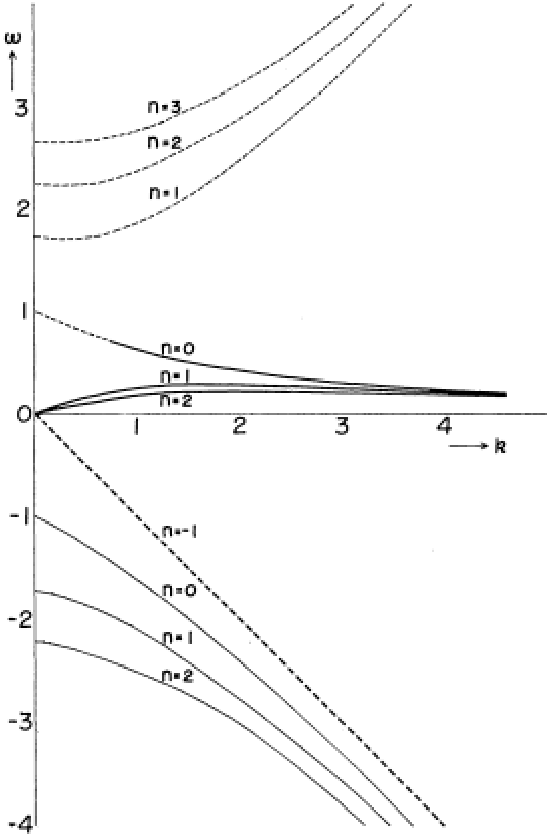
- … With great effort

- Only rotating-shallow-water equations required

  Matsuno, Taroh. (1966). *Quasi-geostrophic motions in the equatorial area.* Journal of the Meteorological Society of Japan. Ser. II 44.1. 25-43.

- Atmospheric observations have since vindicated this work

Kiladis, George N., et al. (2009). *Convectively coupled equatorial waves.* Reviews of Geophysics 47.2.

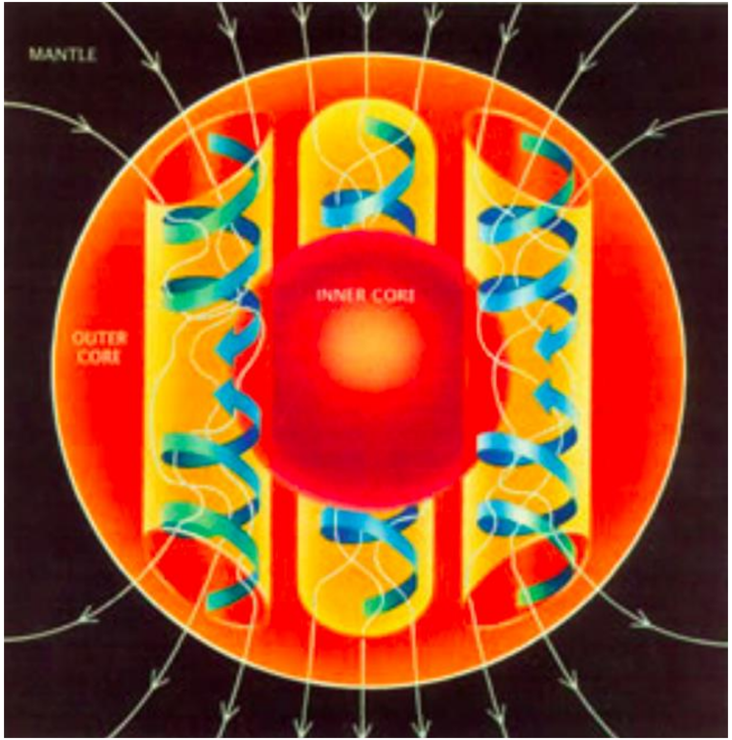


Dispersion relations of various equatorial wave modes

Topological methods have been applied to hydrodynamics

- Mostly in the context of dynamo theory or magnetohydrodynamics
- Protected edge states not considered

Arnold, Vladimir I., and Boris A. Khesin. (1999). *Topological methods in hydrodynamics.* Vol. 125. Springer Science & Business Media.

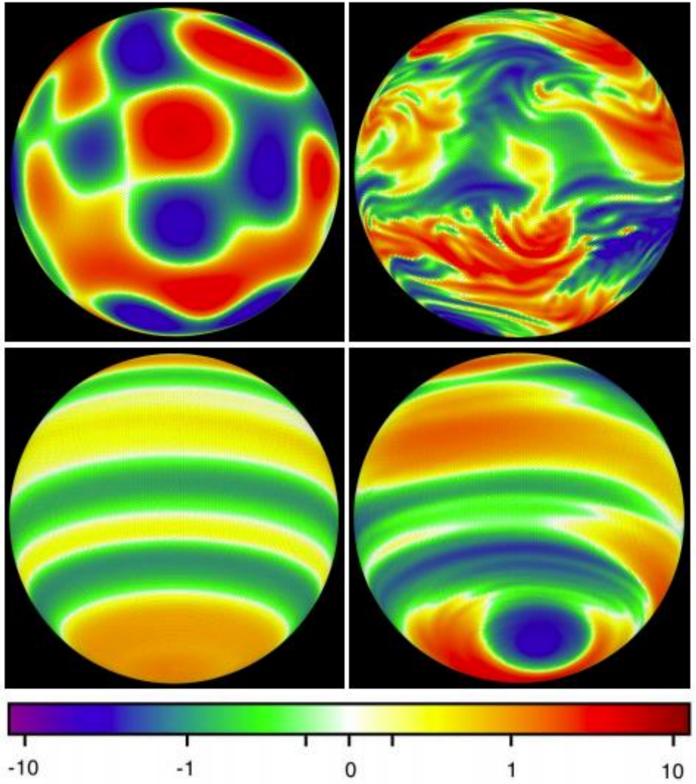


The Earth's mantle is an application of topology to hydrodynamics

Atmospheres have been treated as condensed matter

- Author also co-wrote "Topological origin"
- Built condensed matter models of planetary atmospheres
- Does not discuss topology

Martson, J. B. (2012). *Planetary atmospheres as nonequilibrium condensed matter.*



Topological methods have great predictive power for atmospheric quantities like vorticity

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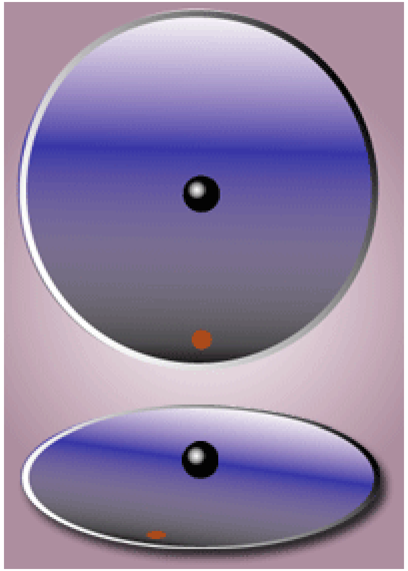
Coriolis force causes equatorial waves

- Ocean and atmospheric waves trapped close to the equator
- Rapid decay away from the equator due to Coriolis force
- Spherical shape of the earth increases the magnitude of the Coriolis force away from the equator

$$\boldsymbol{a}_C = 2\boldsymbol{v} \times \boldsymbol{\Omega}$$ *Coriolis acceleration*

$$\boldsymbol{F}_C = 2m\boldsymbol{v} \times \boldsymbol{\Omega}$$ *Coriolis force*

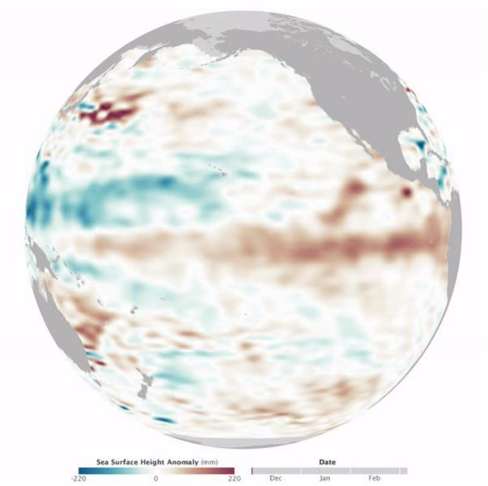


(German Wikipedia)

Coffin, Joseph George David (1911). *Vector Analysis: An Introduction to Vector-methods and Their Various Applications to Physics and Mathematics*. New York: J. Wiley & Sons. p. 198.

Kelvin and Yanai (Rossby-Gravity) waves have been studied previously

- Propagate energy eastward along the equator
- Kelvin modes travel eastward
- Yanai modes can travel westward given periods are substantially long
- Contribute to earth’s climate dynamics
  - El Niño-Southern oscillation
  - Quasi-biennial oscillation in the stratosphere
  - Madden-Julian Oscillation in the troposphere
  - Monsoons

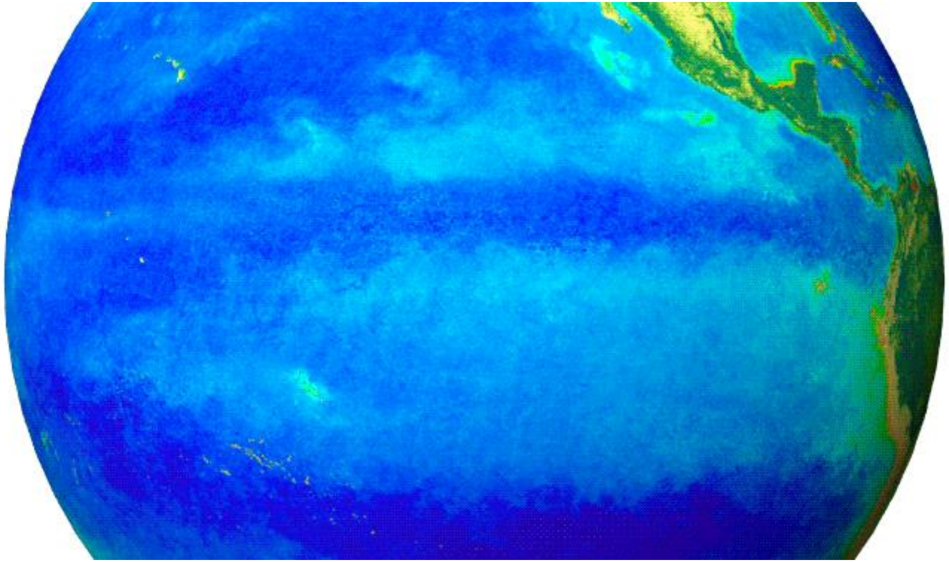


Kelvin Waves

https://earthobservatory.nasa.gov/images/43105/kelvin-wave-renews-el-niao

El Niño-Southern Oscillation is a Kelvin Wave

- ❖ Warm water is transferred across the Pacific to South America
- ❖ Causes extreme weather events
- ❖ Excitations in the Indian ocean excite a Kelvin Wave
- ❖ Kelvin Wave travels across the Pacific in 4 months



Phytoplankton in January immediately after El Niño, and in July

Kelvin and Yanai waves can be derived using shallow-water equations

- β-plane approximation: takes Coriolis parameter to vary linearly in space ($f = \beta y$)

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial h}{\partial y} = 0$$

$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$\boldsymbol{H}$: depth of the fluid

$\boldsymbol{x}$: zonal (horizontal) direction

$\boldsymbol{y}$: meridional direction

$\boldsymbol{u}, \boldsymbol{v}$: $\boldsymbol{x}$ and $\boldsymbol{y}$ fluid velocities

Kelvin and Yanai waves can be derived using shallow-water equations

- Two of the solutions have eastward group velocity

$$u = \hat{u}(y)e^{i(kx+\sigma t)}$$

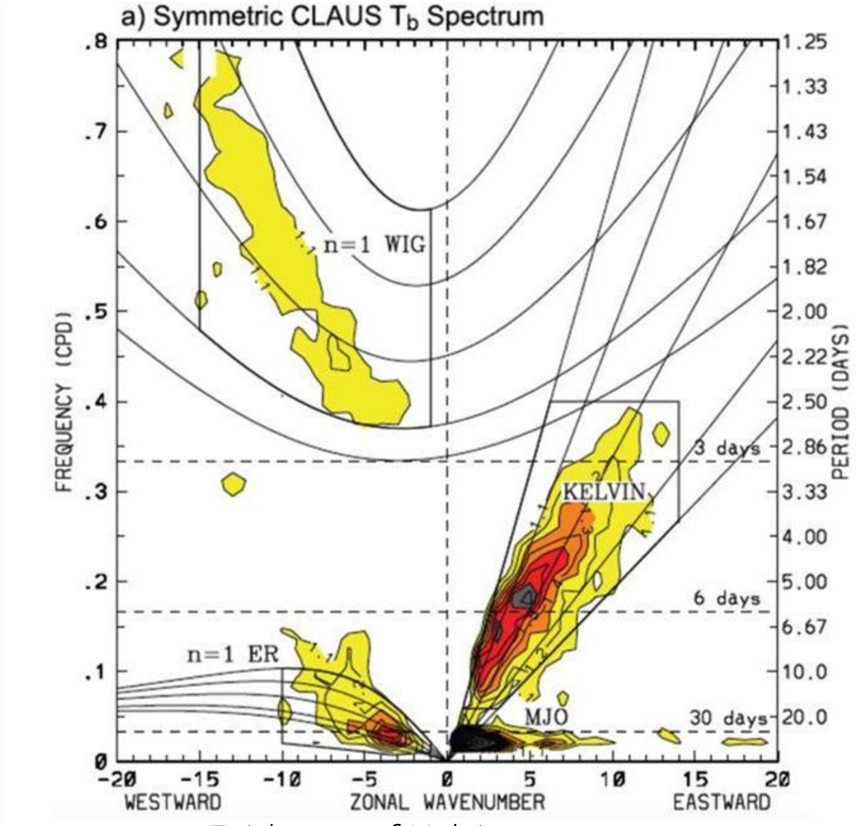
$$v = \hat{v}(y)e^{i(kx+\sigma t)}$$

$$\phi = \hat{\phi}(y)e^{i(kx+\sigma t)}$$

Kelvin: $\sigma = -k$

Yanai: $\sigma = \sqrt{\left(\frac{k}{2}\right)^2 + 1} - \frac{k}{2}$

Kelvin Waves are Observed in the Oceans



Evidence of Kelvin waves

Solving the Wave Equation

$$\partial_t h + \nabla \cdot (h\mathbf{u}) = 0$$
$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -g\nabla h - f\hat{n} \times \mathbf{u}$$

$$f = 2\boldsymbol{\Omega} \cdot \hat{n}$$

Solving the Wave Equation

$$\partial_t h + \nabla \cdot (h\mathbf{u}) = 0$$

$$\boxed{\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\,\mathbf{u}} = -g\nabla h - f\hat{n} \times \mathbf{u}$$

Total time derivative of velocity field (acceleration)

$$f = 2\mathbf{\Omega} \cdot \hat{n}$$

Solving the Wave Equation

$$\partial_t h + \nabla \cdot (h\mathbf{u}) = 0$$
$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = \boxed{-g\nabla h} - f\hat{n} \times \mathbf{u}$$

Force of gravity

$$f = 2\boldsymbol{\Omega} \cdot \hat{n}$$

Solving the Wave Equation

$$\partial_t h + \nabla \cdot (h\mathbf{u}) = 0$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -g\nabla h \boxed{- f\hat{n} \times \mathbf{u}}$$

Coriolis force

$$\boxed{f = 2\boldsymbol{\Omega} \cdot \hat{n}}$$

Finding Bulk Solutions

- Linearizing the equations gives a Schrödinger Equation
- The planewave solutions satisfy a dispersion relation

$$i\partial_t \Psi = H\Psi$$

$$\Psi = \begin{pmatrix} u_x \\ u_y \\ h - h_0 \end{pmatrix}$$

$$\Psi = \Psi_0 \mathrm{e}^{i(\omega t - k_x x - k_y y)}$$

$$\omega = \pm\sqrt{f^2 + c^2\mathbf{k}^2}$$

Finding Solutions

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Coriolis effect breaks time-reversal invariance, causing the gap in the spectrum

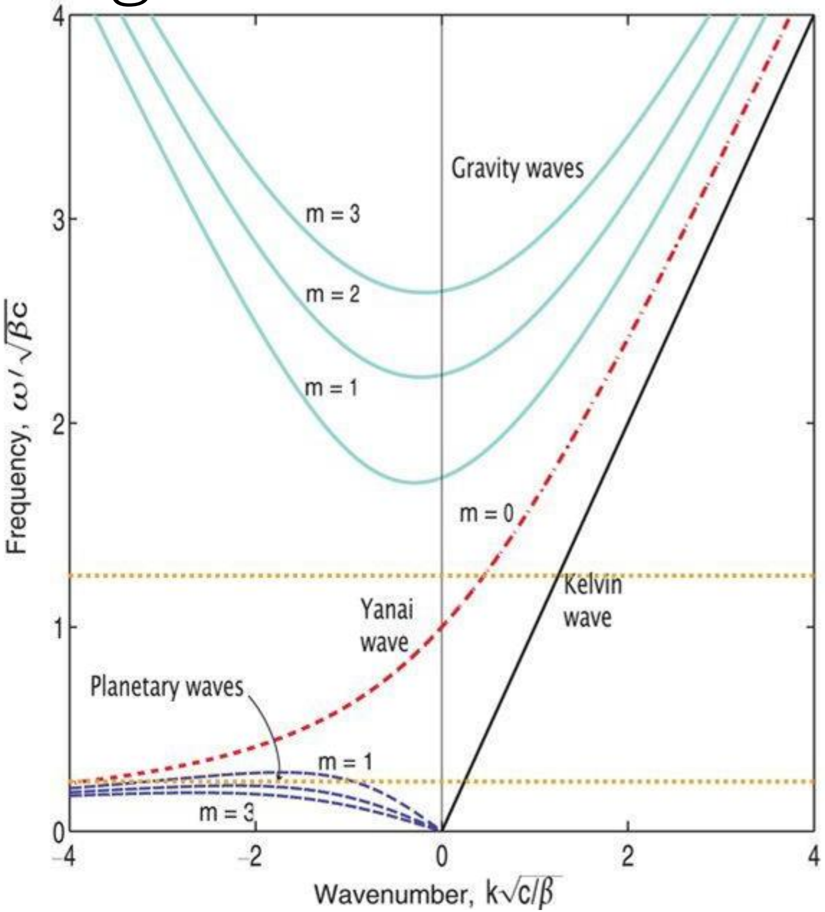
Topologically protected edge states are nothing new

- Predicted in 1987, physically realized in 2008
- Bulk-boundary correspondence well established and not controversial
- Still somewhat popular to this day

Charles Kane and Joel Moore (2011). *Phys. World* **24** (02) 32

Patching the bulk solutions together

- Planewave solutions are good on a patch of the sphere with constant coriolis parameter $f = 2\mathbf{\Omega} \cdot \hat{n}$
- Full solution patches together solutions around the sphere
- This can’t be done consistently because the system has Chern number = 2
- Implies existence of two edge modes at the equator



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How does “Topological origin” compare to previous work?

- Applies ideas from topology in condensed matter to geophysics in a novel way
- Crucially exploits the bulk-boundary correspondence
- Successfully replicates vetted results from classical theory and observation

Our impressions

- Positives
  - Novel application of techniques from condensed matter physics to other areas
  - Advertisement for topological insulators

- Negatives
  - There are exactly three equatorial wave modes
  - Analysis of real wave functions could elucidate the mechanism that prefers eastward over westward group velocity
  - Difficult to generalize to other fields

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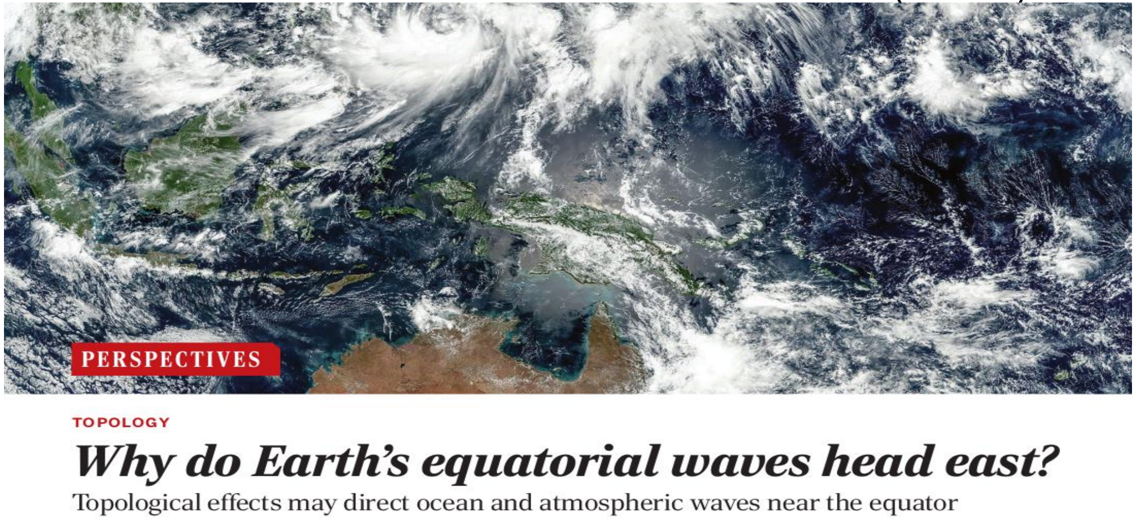
Citation Evaluation

- The paper is cited by 10 documents
- "Intrinsic Pink-Noise Multidecadal Global Climate Dynamics Mode." ***Physical review letters*** 121.10 (2018): 108701.
- "Why do Earth's equatorial waves head east?." ***Science*** 358.6366 (2017): 990-991.



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Authors’ Conclusions

- The notion of topologically protected edge states extends naturally to oceanic waves
- Observed dispersion relations were replicated with ideas from topology
- These techniques can be generalized to even more hydrodynamical systems

Our conclusions

- Positives
  - Innovative extension of condensed matter physics
  - Serves as justification for further study of topological insulators
- Negatives
  - Extensive work remains to make the result robust
  - Generalizations to other systems may not be as straightforward as authors purport

Questions?