Problem 1 (20 points)

How does $\bar{\psi}\sigma^{\mu\nu}\psi$ transform under proper Lorentz transformation? How does each of the six components of $\bar{\psi}\sigma^{\mu\nu}\psi$ transform under space inversion?

Problem 2 (25 points)

a) The operators

$$P_R \equiv \frac{1}{2}(1 + \gamma^5), \quad P_L \equiv \frac{1}{2}(1 - \gamma^5)$$

are the right-hand and left-hand projection operators. Show that they have the appropriate properties to be projection operators, namely,

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L + P_R = 1, \quad P_L P_R = 0.$$ 

b) The operators to project out the positive and negative energy states for the Dirac spinors are

$$\Lambda_+ \equiv \frac{\gamma^\mu + m}{2m}, \quad \Lambda_- \equiv -\frac{\gamma^\mu - m}{2m}.$$ 

Show that they have the appropriate properties to be projection operators, namely,

$$\Lambda_+^2 = \Lambda_-, \quad \Lambda_-^2 = \Lambda_+, \quad \Lambda_+ + \Lambda_- = 1, \quad \Lambda_+ \Lambda_- = 0.$$ 

Problem 3 (25 points)

In terms of a four-component spinor $\psi$, the right- and left-handed helicity states are defined as

$$\psi_R = \frac{1}{2}(1 + \gamma_5)\psi,$$

$$\psi_L = \frac{1}{2}(1 - \gamma_5)\psi.$$ 

Show that

$$\bar{\psi}\gamma_\mu\psi = \bar{\psi}_L\gamma_\mu\psi_L + \bar{\psi}_R\gamma_\mu\psi_R,$$

$$\bar{\psi}\gamma_\mu\gamma_5\psi = \bar{\psi}_L\gamma_\mu\gamma_5\psi_L + \bar{\psi}_R\gamma_\mu\gamma_5\psi_R,$$

$$\bar{\psi}m\psi = \bar{\psi}_Lm\psi_R + \bar{\psi}_Rm\psi_L.$$ 

This exercise shows that the vector and axial vector currents do not connect spinors of different chirality, while the mass $m$ (or scalar potential) flips helicity.
Problem 4 (30 points)

Define

\[ \psi_L = \frac{1 - \gamma_5}{2} \psi; \quad \psi_R = \frac{1 + \gamma_5}{2} \psi \]

a) Show that the charge conjugate of \( \psi_L \) is right-handed and the charge conjugate of \( \psi_R \) is left-handed.

b) Under time-reversal transformation, what are the handedness of \( \psi'_L \) and \( \psi'_R \)?

c) Under parity transformation, what are the handedness of \( \psi'_L \) and \( \psi'_R \)?