Chapter 3

Building Hadrons from Quarks

Mesons in SU(2)

We are now ready to consider mesons and baryons constructed from quarks. As we recall, mesons are made of quark-antiquark pair and baryons are made of three quarks.

Consider mesons made of u and d quarks first.

$$\psi = \psi\text{ (space) }\psi\text{ (spin) }\psi\text{ (flavor) }\psi\text{ (color)}$$

It is straight-forward to write down the spin wave function of a meson. Since both q and $\bar{q}$ have spin ½, the $q\bar{q}$ can form either a spin-triplet or a spin-singlet state:

$$\left.\begin{aligned} |1,1\rangle &= \uparrow\uparrow \\ |1,0\rangle &= \frac{1}{\sqrt{2}}\left(\uparrow\downarrow + \downarrow\uparrow\right) \\ |1,-1\rangle &= \downarrow\downarrow \end{aligned}\right\} \text{ spin 1 triplet}$$

$$\left. |0,0\rangle = \frac{1}{\sqrt{2}}\left(\uparrow\downarrow - \downarrow\uparrow\right) \quad \right\} \text{ spin 0 singlet}$$

For the flavor wave function, we have an analogous situation. The u, d quarks are isospin doublet, and so are $\bar{d}$, $\bar{u}$. However, a slight complication arises since $\bar{u}$, $\bar{d}$ transforms in the complex conjugate representation.

Now $\begin{pmatrix} u' \\ d' \end{pmatrix} = I_i \begin{pmatrix} u \\ d \end{pmatrix}$ and $\left[I_i,\, I_j\right] = i\ \varepsilon_{ijk} I_k$

taking complex conjugate, we get $\left[I_i^*,\, I_j^*\right] = -i\ \varepsilon_{ijk} I_k^*$

or $\left[-I_i^*,\, -I_j^*\right] = i\ \varepsilon_{ijk}\left(-I_k^*\right)$

$-I_i^*$ transforms $\bar{u}$, $\bar{d}$ into $\bar{u}'$, $\bar{d}'$:

$$\begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} = \left(-I_i^*\right)\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \qquad \text{(Eq. 1)}$$

write out $-I_i^*$ explicitly:

$$\left(-I_1^*\right) = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \left(-I_2^*\right) = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \left(-I_3^*\right) = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

One can easily verify that $-I_i^*$ is related to I_i by a unitary transformation

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(-I_i^*\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = I_i$$

One can multiply (Eq. 1) by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on the left

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(-I_i^*\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{I_i}\!\!\overbrace{\phantom{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}}^{1}\!\!\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \qquad \text{(Eq. 2)}$$

(Eq. 2) becomes
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} = I_i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}$$

or
$$\begin{pmatrix} -\bar{d}' \\ \bar{u}' \end{pmatrix} = I_i\begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$$

Therefore, $\begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$ transforms just like $\begin{pmatrix} u \\ d \end{pmatrix}$ and is the isospin doublet for $\bar{u}$, $\bar{d}$.

Combining the $\begin{pmatrix} u \\ d \end{pmatrix}$ with the $\begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$, one obtains isospin triplet and isospin singlet:

$$\left.\begin{aligned} |I, I_z\rangle &= |1,1\rangle = -u\bar{d} \\ |1,0\rangle &= \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right) \\ |1,-1\rangle &= \bar{u}d \end{aligned}\right\} \quad I = 1$$

$$\left.|0,0\rangle = \frac{1}{\sqrt{2}}\left(u\bar{u} + d\bar{d}\right)\right\} \quad I = 0$$

The total spin of the mesons can be $S = 0$ or 1. The lightest mesons have orbital angular momentum $L = 0$. Therefore, their total angular momentum $J = 0$ or 1.

$J = 0, I = 1$	π^+, π^o, π^-	(mass ~ 140 MeV)
$J = 1, I = 1$	ρ^+, ρ^o, ρ^-	(mass ~ 770 MeV)
$J = 1, I = 0$	ω^o	(mass ~ 780 MeV)

(We will postpone the discussion of $J = 0$, $I = 0$ meson until later, since its wave function contains non-u, d components.)

One can readily combine the flavor and spin parts of the wavefunction. For example:

$$\rho^o\left(J_z = 0\right) = \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right)\frac{1}{\sqrt{2}}\left(\uparrow\downarrow + \downarrow\uparrow\right) = \frac{1}{2}\left(u\uparrow\bar{u}\downarrow + u\downarrow\bar{u}\uparrow - d\uparrow\bar{d}\downarrow - d\downarrow\bar{d}\uparrow\right)$$

$$\pi^+\left(J_z = 0\right) = -u\bar{d}\,\frac{1}{\sqrt{2}}\left(\uparrow\downarrow - \downarrow\uparrow\right) = -\frac{1}{\sqrt{2}}\left(u\uparrow\bar{d}\downarrow - u\downarrow\bar{d}\uparrow\right)$$

<u>Baryons in SU(2)</u>

Now, consider baryons made of u, d quarks. Let us start with the flavor part.

First, combining two quarks:

$$\left.\begin{array}{l} |1,1\rangle \ : \ uu \\ |1,0\rangle \ : \ \frac{1}{\sqrt{2}}(ud+du) \\ |1,-1\rangle \ : \ dd \end{array}\right\} I=1 \text{ triplet}$$

$$\left.|0,0\rangle \ : \ \frac{1}{\sqrt{2}}(ud-du)\right\} I=0 \text{ singlet}$$

Note that $I = 1$ is symmetric with respect to the interchange of the two quarks, while $I = 0$ singlet is anti-symmetric. We therefore label the symmetry property of the representations as

$$2\otimes 2 = 3_S \oplus 1_A$$

where the dimension of the representation is $2I+1$ (For example, I=1/2 →2; I=1 →3; I=0 →1). Adding another u/d quark, we have

$$2\otimes 2\otimes 2 = (3_S + 1_A)\otimes 2 = (3_S\otimes 2)\oplus(1_A\otimes 2)$$

First consider $1_A\otimes 2$:

$$\frac{1}{\sqrt{2}}(ud-du)\otimes\begin{pmatrix} u \\ d \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}}(ud-du)u \\ \frac{1}{\sqrt{2}}(ud-du)d \end{pmatrix}$$

This is an isospin doublet with mixed permutation symmetry (anti-symmetric WRT the interchange of the first two quarks). Therefore, we denote

$$1_A\otimes 2 = 2_{M_A}$$

where M signifies 'mixed-symmetry' while the subscript A reminds us the anti-symmetry between the first two quarks.

We now consider the remaining part, $3_S \otimes 2$:

$$\begin{pmatrix} uu \\ \frac{1}{\sqrt{2}}(ud+du) \\ dd \end{pmatrix} \otimes \begin{pmatrix} u \\ d \end{pmatrix} \qquad \left(I=1 \otimes I = \tfrac{1}{2} \Rightarrow I = \tfrac{3}{2} \oplus I = \tfrac{1}{2}\right)$$

This simply corresponds to the product of an isospin triplet and an isospin doublet. The resulting isospin can have $I = \frac{3}{2}$ and $I = \frac{1}{2}$.

$I = \frac{3}{2}$:

$$\begin{pmatrix} uuu \\ \frac{1}{\sqrt{3}}uud + \sqrt{\frac{2}{3}}\frac{1}{\sqrt{2}}(ud+du)u \\ \sqrt{\frac{2}{3}}\frac{1}{\sqrt{2}}(ud+du)d + \sqrt{\frac{1}{3}}ddu \\ ddd \end{pmatrix} = \begin{pmatrix} uuu \\ \frac{1}{\sqrt{3}}(uud+udu+duu) \\ \frac{1}{\sqrt{3}}(udd+dud+ddu) \\ ddd \end{pmatrix}$$

$I = \frac{1}{2}$:

$$\begin{pmatrix} \frac{\sqrt{2}}{3}uud - \sqrt{\frac{1}{3}}\frac{1}{\sqrt{2}}(ud+du)u \\ \frac{\sqrt{2}}{3}\frac{1}{\sqrt{2}}(ud+du)d - \sqrt{\frac{2}{3}}ddu \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}}(2uud-udu-duu) \\ \frac{1}{\sqrt{6}}(udd+dud-2ddu) \end{pmatrix}$$

In constructing the $I = \frac{3}{2}$ and $I = \frac{1}{2}$ multiplets, we simply use the Clebsch-Gordon coefficients to combine the $\left|I_1\ I_{1z}\ I_2\ I_{2z}\right\rangle$ states into $\left|I\ \ I_z\right\rangle$ states.

$I = \frac{3}{2}$ multiplet is totally symmetric WRT interchange of any pair, while $I = \frac{1}{2}$ multiplet has mixed-symmetry and is symmetric WRT the interchange of the first two quarks.

Therefore, we write $$3_S \otimes 2 = 4_S \oplus 2_{M_S}$$

Collecting previous result, we now have $$2 \otimes 2 \otimes 2 = 4_S \oplus 2_{M_S} \oplus 2_{M_A}$$

The spin wave functions for three quarks are constructed exactly the same way.

The flavor-spin wave functions of the $\Delta^{++}, \Delta^{+}, \Delta^{o}, \Delta^{-}$ quartet are constructed by combining the $I = \frac{3}{2}$ (flavor) with the $J = \frac{3}{2}$ (spin) wave function.

$$\Delta: (\underset{\text{flavor}}{4_S},\ \underset{\text{spin}}{4_S})$$

For example $\Delta^{++}\left(J_Z = \frac{3}{2}\right)$ is simply $(uuu)(\uparrow\uparrow\uparrow) = u\uparrow u\uparrow u\uparrow$

while $\Delta^{+}\left(J_Z = \frac{1}{2}\right)$ is

$$\frac{1}{\sqrt{3}}(uud + udu + duu)\frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$$
$$= \frac{1}{3}\left[\left(u\uparrow u\uparrow d\downarrow + u\uparrow u\downarrow d\uparrow + u\downarrow u\uparrow d\uparrow\right) + \text{ permutations}\right]$$

Note that the Δ flavor-spin wave function is totally symmetric with respect to exchange of any pair of quarks. Since quarks are fermions, the wave function should be anti-symmetric overall. We now believe the color wave function is anti-symmetric (in order to become a color-singlet state), making the full wave function anti-symmetric.

The proton and neutron, being spin-½ isospin-½ particles, are constructed with the following combination to make the overall flavor-spin wave function symmetric.

$$\sqrt{\frac{1}{2}}\left[\left(\underset{\text{flavor}}{2_{M_S}},\ \underset{\text{spin}}{2_{M_S}}\right)+\left(\underset{\text{flavor}}{2_{M_A}},\ \underset{\text{spin}}{2_{M_A}}\right)\right]$$

As an example, consider proton with $J_Z = ½$:

$$p\left(J_Z = 1/2\right) = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{6}}(2uud - udu - duu) \bullet \frac{1}{\sqrt{6}}\left(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\right)\right.$$
$$\left. + \frac{1}{\sqrt{2}}(udu - duu)\frac{1}{\sqrt{2}}\left(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\right)\right]$$
$$= \frac{1}{\sqrt{18}}\left[uud\left(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\right) + udu\left(2\uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow\right)\right.$$
$$\left. + duu\left(\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow\right)\right]$$
$$= \frac{1}{\sqrt{18}}\left\{\left[2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow\right] + \text{permutations}\right\}$$

Note that p (J_Z = - ½) can be obtained from p (J_Z = + ½) by interchanging ↑ with ↓.

Similarly, n (J_Z = ½) can be obtained from p (J_Z = ½) by interchanging u with d quarks.

SU(3)

Adding strange quark, we extend SU(2) to SU(3):

$$\begin{pmatrix} u \\ d \end{pmatrix} \Rightarrow \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

SU(N) has N^2-1 generators, and the fundamental representation has matrices of dimension N. The eight generators for SU(3) satisfy the Lie algebra:

$$\left[T_i,\ T_j\right] = i\, f_{ijk}\ T_k$$

where the structure constants are

$$f_{123} = 1,\ f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = {}^{1}\!/_{2},\ f_{458} = f_{678} = {}^{\sqrt{3}}\!/_{2}$$

The structure constants are anti-symmetric with respect to interchange of any two indices.

The fundamental representations for SU(3) are:

$$T_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad T_2 = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad T_4 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$T_5 = \frac{1}{2}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \qquad T_6 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

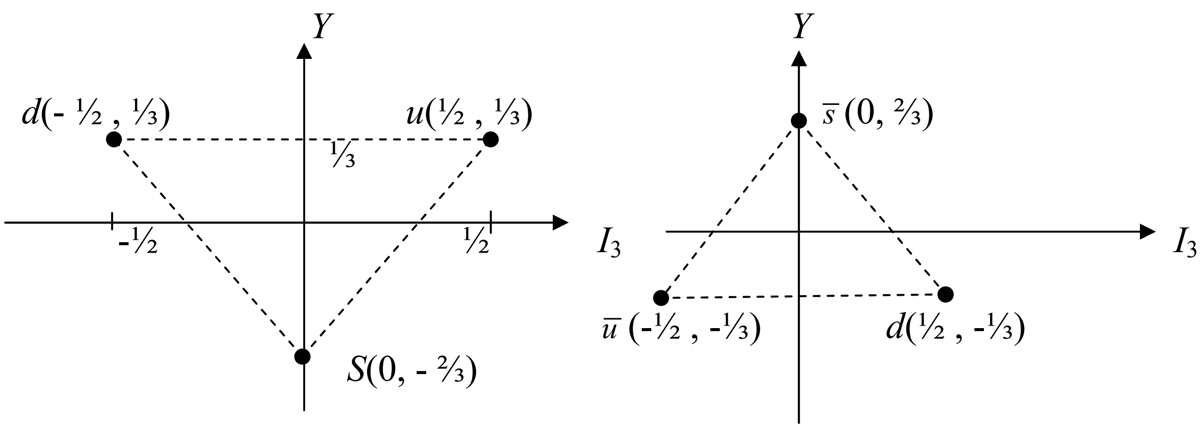
$$T_7 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad T_8 = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

<u>Mesons in SU(3)</u>

u, *d*, *s* quarks belong to the fundamental representation of SU(3), denoted by 3. $\overline{u},\ \overline{d},\ \overline{s}$ belong to the conjugate representation $\overline{3}$. Combining quark and anti-quark together:

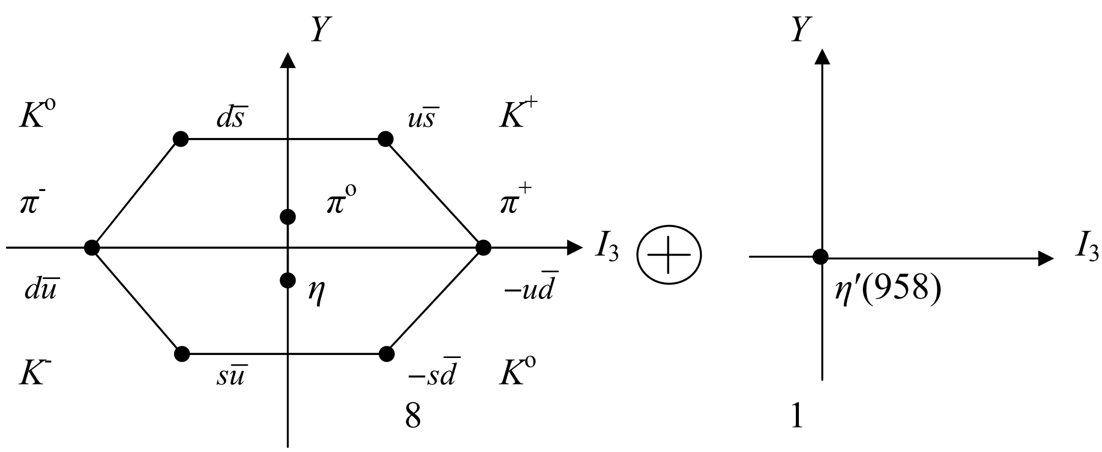
$$3 \otimes \overline{3} = 8 \oplus 1$$

In terms of the 'weight' diagram, the 3 and $\overline{3}$ representations are:



Where $Y = B + S$, B is the baryon number and S is the strangeness.

$3 \otimes \overline{3}$ is obtained by superimposing the $\overline{3}$ triplet on top of each site of the quark 3 triplet. One obtains



The SU(3) singlet must contain s, u, d quarks equally and has the wave function

$$\sqrt{\frac{1}{3}}\left(u\overline{u} + d\overline{d} + s\overline{s}\right)$$

The other two mesons occupying the $I_3 = 0$, Y = 0 location are the SU(2) triplet state

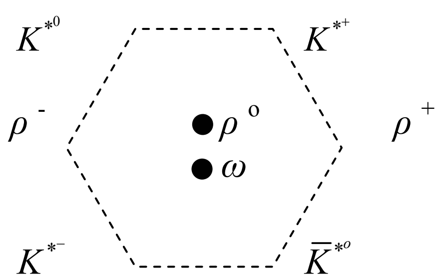
$$\frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right)$$

and the SU(2) singlet state

$$\frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right)$$

The lightest meson octet consists of $K^+\left(u\bar{s}\right)$, $K^o\left(d\bar{s}\right)$, $\pi^+\left(-u\bar{d}\right)$, $\pi^o\left(\frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right)\right)$, $\pi^-\left(d\bar{u}\right)$ and $\eta\left(\frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right)\right)$. Together with the SU(3) singlet (η'), they form the pseudoscalar (0^-) nonet.

The $J = 1$ meson octet consists of:



and the $J = 1$ singlet is φ.

The ω and φ mesons are found to be mixtures of the SU(3) states ω_8, φ_1

$$\varphi \simeq -\sqrt{\frac{2}{3}}\ \omega_8 + \sqrt{\frac{1}{3}}\ \varphi_1 = s\bar{s}$$
$$\omega \simeq \sqrt{\frac{1}{3}}\ \omega_8 + \sqrt{\frac{2}{3}}\ \varphi_1 = \frac{1}{\sqrt{2}}\left(u\bar{u} + d\bar{d}\right)$$

Evidence for φ to be nearly a pure $s\bar{s}$ state is from the peculiar behavior of φ decays.

φ has a width of only 4.3 MeV and its relatively long lifetime is due to the fact that the main decay channels for φ are

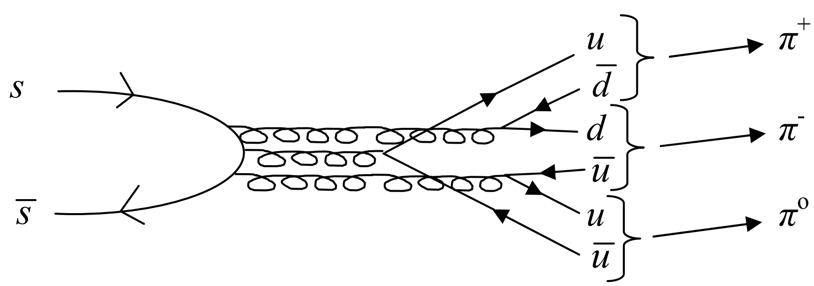
$$\varphi \to K^+K^-,\ K_L^oK_s^o \qquad \text{(Branching Ratio } \simeq \text{ 83\%)}$$

The phase space for φ (1020) decaying into a pair of kaons is greatly suppressed relative to the

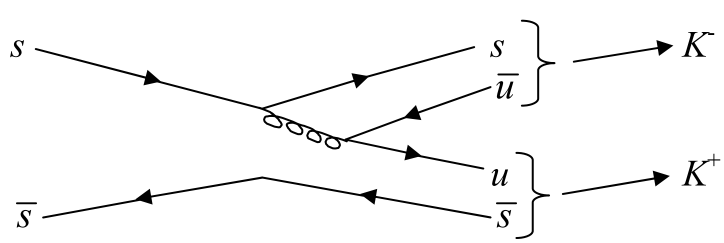
$$\varphi \to \pi^+\pi^-\pi^o \qquad \text{(Branching Ratio } \simeq \text{ 16\%)}$$

However, the $\varphi \to 3\pi$ decay is suppressed by the Zweig rule if φ is mainly a $s\overline{s}$ state. The $\varphi \to KK$ decays, which are not affected by the Zweig rule, became the dominant decay channels.

The Zweig rule (or OZI rule, due to Okubo, Zweig, and Iizuka) states that decay rates for processes described by diagrams with unconnected quark lines are suppressed. Thus, $\varphi \to \pi^+\pi^-\pi^o$ decay diagram



shows that the 3π quark lines are completely disconnected from the initial $s\overline{s}$ quark lines. In contrast, the $\varphi \to K^+K^-$ decay diagram has some quark lines connected, and it is not suppressed by the Zweig rule.

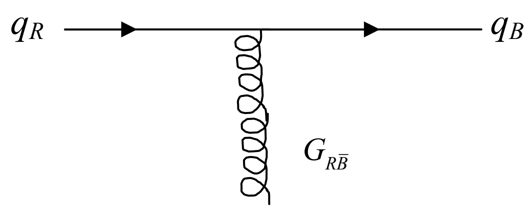


The three colors (R, B, G) of quark transform under the SU(3) exact symmetry. Since mesons are made of $q-\bar{q}$, their color wave function should be an SU(3)$_c$ singlet. The singlet state in $3_c \otimes \bar{3}_c = 8_c \oplus 1_c$ is simply

$$\frac{1}{\sqrt{3}}\left(R\bar{R}+B\bar{B}+G\bar{G}\right)$$

Therefore, the color wave functions of all mesons have the same form as the flavor wave function of an SU(3) singlet meson (η').

As discussed earlier, the color contents of gluons are (color x $\overline{\text{color}}$). Recall the quark-gluon coupling diagram:



for a 'R' quark changing into a 'B' quark and the gluon carrying $R\bar{B}$ color. The gluons therefore have eight different color combinations:

$$R\bar{G},\ R\bar{B},\ G\bar{R},\ G\bar{B},\ B\bar{R},\ B\bar{G},\ \sqrt{\frac{1}{2}}\left(R\bar{R}-G\bar{G}\right),\ \sqrt{\frac{1}{6}}\left(R\bar{R}+G\bar{G}-2B\bar{B}\right)$$

Note that the color contents of the 8 gluons are analogous to the flavor contents of the meson octet.

<u>Baryons in SU(3)</u>

The spin wave functions for baryons have been discussed earlier:

$$2\otimes 2\otimes 2=\left(3\oplus 1\right)\otimes 2=4_s\oplus 2_{M_S}\oplus 2_{M_A}$$

For the flavor wave function, we first consider $3\otimes 3$

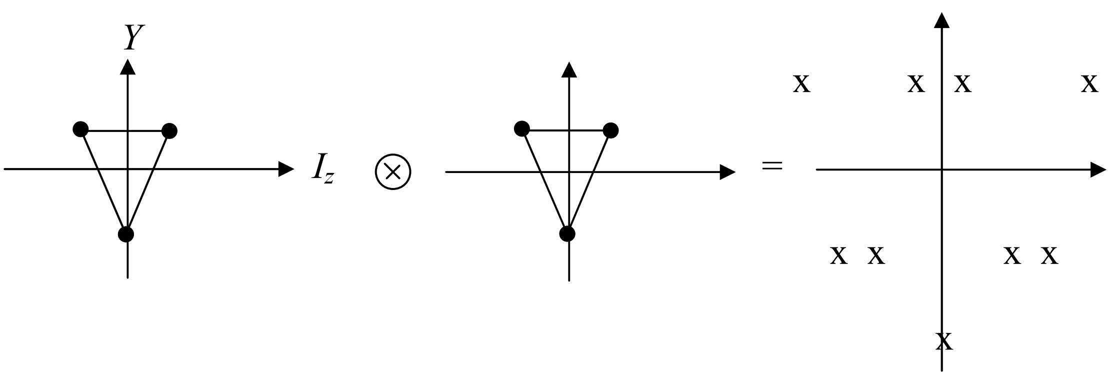
$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \otimes \begin{pmatrix} u \\ d \\ s \end{pmatrix} \text{ gives 9 basis states:} \qquad uu, ud, us, du, dd, ds, su, sd, ss$$

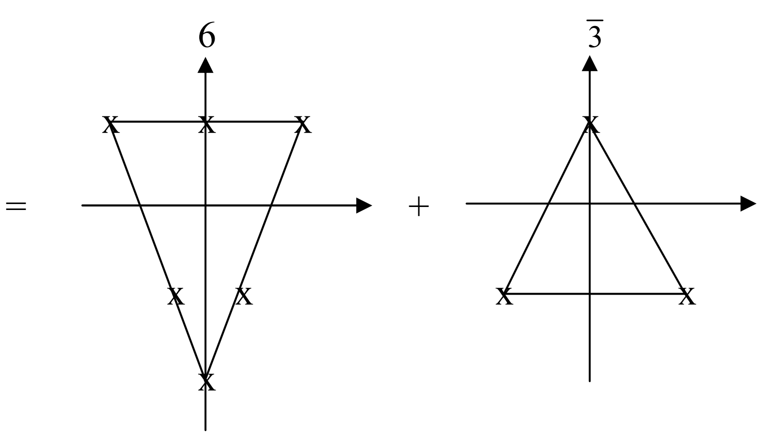
These basis states can be rearranged to form basis states with specific permutation symmetry. In particular, the basis states $\psi_i \varphi_j \left(i = u,\ d,\ s;\ j = u,\ d,\ s\right)$ can be expressed as

$$\psi_i \varphi_j = \underbrace{\frac{1}{2}\left(\psi_i \varphi_j + \psi_j \varphi_i\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left(\psi_i \varphi_j - \psi_j \varphi_i\right)}_{\text{anti-symmetric}} = S_{ij} + A_{ij}$$

It is clear that the symmetric S_{ij} contains 6 elements, while the anti-symmetric A_{ij} contains 3 elements $\quad 3 \otimes 3 = 6_s \oplus \overline{3}_A$

In terms of the SU(3) weight diagram





Now $3 \otimes 3 \otimes 3 = \left(6_s \oplus \overline{3}_A\right) \otimes 3 = \left(6_s \otimes 3\right) \oplus \left(\overline{3}_A \otimes 3\right)$

First consider $6_s \otimes 3$

$$S_{ij}\psi_k = \frac{1}{2}\left(S_{ij}\psi_k + S_{jk}\psi_i + S_{ki}\psi_j\right) + \frac{1}{2}\left(S_{ij}\psi_k - S_{jk}\psi_i - S_{ki}\psi_j\right)$$

The first part of the decomposition is totally symmetric and contains 10 elements, while the second part contains 8 elements and has mixed symmetry (symmetric only for $i \leftrightarrow j$ interchange).

Therefore $6_s \otimes 3 = 10_s \oplus 8_{M_S}$

Next consider $\overline{3}_A \otimes 3$

$$A_{ij}\psi_k = \frac{1}{2}\left(A_{ij}\psi_k + A_{jk}\psi_i + A_{ki}\psi_j\right) + \frac{1}{2}\left(A_{ij}\psi_k - A_{jk}\psi_i - A_{ki}\psi_j\right)$$

The first term has 1 element and is totally anti-symmetric, while the second term has 8 elements with mixed symmetry (anti-symmetric only for $i \leftrightarrow j$ interchange).

Therefore $\overline{3}_A \otimes 3 = 8_{M_A} \oplus 1_A$

Finally we have $3 \otimes 3 \otimes 3 = 10_s \oplus 8_{M_S} \oplus 8_{M_A} \oplus 1_A$

The explicit flavor wave functions for 10_s, 8_{M_S}, 8_{M_A}, and 1_A are listed on the next two pages.

SU(3) Baryons

Flavor: $$3\otimes 3\otimes 3 = 10_s \oplus 8_{M_S} \oplus 8_{M_A} \oplus 1_A$$

10_s:

Δ^{++} : uuu

Δ^{+} : $\frac{1}{\sqrt{3}}(uud + udu + duu)$

Δ^{o} : $\frac{1}{\sqrt{3}}(ddu + dud + udd)$

Δ^{-} : ddd

Σ^{+^*} : $\frac{1}{\sqrt{3}}(uus + usu + suu)$

Σ^{o^*} : $\frac{1}{\sqrt{6}}(uds + usd + dsu + dus + sud + sdu)$

Σ^{-^*} : $\frac{1}{\sqrt{3}}(dds + dsd + sdd)$

Ξ^{o^*} : $\frac{1}{\sqrt{3}}(ssu + sus + uss)$

Ξ^{-^*} : $\frac{1}{\sqrt{3}}(ssd + sds + dss)$

Ω^{-^*} : sss

8_{M_S} :

p : $\frac{1}{\sqrt{6}}(2uud - udu - duu)$

n : $\frac{1}{\sqrt{6}}(-2ddu + dud + udd)$

Σ^{+} : $\frac{1}{\sqrt{6}}(2uus - usu - suu)$

Σ^{o} : $\frac{1}{\sqrt{12}}(2uds - usd - dsu + 2dus - sud - sdu)$

Σ^{-} : $\frac{1}{\sqrt{6}}(2dds - dsd - sdd)$

Λ^{o} : $\frac{1}{2}(usd + sud - sdu - dsu)$

Ξ^{o} : $\frac{1}{\sqrt{6}}(sus + uss - 2ssu)$

Ξ^{-} : $\frac{1}{\sqrt{6}}(sds + dss - 2ssd)$

8_{M_A} :

$p: \frac{1}{\sqrt{2}}(udu - duu)$

$n: \frac{1}{\sqrt{2}}(udd + dud)$

$\Sigma^+: \frac{1}{\sqrt{2}}(usu - suu)$

$\Sigma^o: \frac{1}{2}(usd + dsu - sdu - sud)$

$\Sigma^-: \frac{1}{\sqrt{2}}(dsd - sdd)$

$\Lambda^o: \frac{1}{\sqrt{12}}(2uds - dsu - sud - 2dus - sdu + usd)$

$\Xi^o: \frac{1}{\sqrt{2}}(uss - sus)$

$\Xi^-: \frac{1}{\sqrt{2}}(dss - sds)$

1_A: $\frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)$

Note 8_{M_S} is obtained by the following sequence:

starting from $q_1\, q_2\, q_3$

a) symmetrize in positions 1 and 2
b) anti-symmetrize in positions 1 and 3
c) symmetrize in positions 1 and 2

For Λ^o, orthogonality to Σ^o, Σ^{o^*}, and 1_A are imposed to obtain its wave function.

For 8_{M_A}, the procedure is analogous, but with anti-symmetrization in steps a) and c) and symmetrization in step b). Also, Σ^o is obtained by requiring orthogonality to Λ^o, Σ^{o^*} and 1_A.

Spin: $2 \otimes 2 \otimes 2 = 4_s \oplus 2_{M_S} \oplus 2_{M_A}$

4_s:

$$\left|{}^3\!/_2, {}^3\!/_2\right\rangle: \quad \uparrow\uparrow\uparrow$$
$$\left|{}^3\!/_2, {}^1\!/_2\right\rangle: \quad \frac{1}{\sqrt{3}}\left(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow\right)$$
$$\left|{}^3\!/_2, -{}^1\!/_2\right\rangle: \quad \frac{1}{\sqrt{3}}\left(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow\right)$$
$$\left|{}^3\!/_2, -{}^3\!/_2\right\rangle: \quad \downarrow\downarrow\downarrow$$

2_{M_S} :

$$\left|{}^1\!/_2, {}^1\!/_2\right\rangle: \quad \frac{1}{\sqrt{6}}\left(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\right)$$
$$\left|{}^1\!/_2, -{}^1\!/_2\right\rangle: \quad \frac{1}{\sqrt{6}}\left(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow - 2\downarrow\downarrow\uparrow\right)$$

2_{M_A} :

$$\left|{}^1\!/_2, {}^1\!/_2\right\rangle: \quad \frac{1}{\sqrt{2}}\left(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\right)$$
$$\left|{}^1\!/_2, -{}^1\!/_2\right\rangle: \quad \frac{1}{\sqrt{2}}\left(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow\right)$$

Baryon decuplet: $(10_s, 4_s)$

Baryon octet: $\frac{1}{\sqrt{2}}\left[\left(8_{M_S},\ 2_{M_S}\right) + \left(8_{M_A},\ 2_{M_A}\right)\right]$

We can write down explicitly the flavor-spin wave functions of the baryon octet:

$$p\uparrow = \frac{1}{\sqrt{18}}\left\{\left[2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow\right] + \text{permutations}\right\}$$
$$n\uparrow = \frac{1}{\sqrt{18}}\left\{\left[-2d\uparrow d\uparrow u\downarrow + d\uparrow d\downarrow u\uparrow + d\downarrow d\uparrow u\uparrow\right] + \text{permutations}\right\}$$
$$\Sigma^+\uparrow = \frac{1}{\sqrt{18}}\left\{\left[2u\uparrow u\uparrow s\downarrow - u\uparrow u\downarrow s\uparrow - u\downarrow u\uparrow s\uparrow\right] + \text{permutations}\right\}$$
$$\Sigma^o\uparrow = \frac{1}{6}\left\{\left[2u\uparrow d\uparrow s\downarrow - u\uparrow d\downarrow s\uparrow - u\downarrow d\uparrow s\uparrow\right] + \text{permutations}\right\}$$
$$\Sigma^-\uparrow = \frac{1}{\sqrt{18}}\left\{\left[2d\uparrow d\uparrow s\downarrow - d\uparrow d\downarrow s\uparrow - d\downarrow d\uparrow s\uparrow\right] + \text{permutations}\right\}$$
$$\Xi^o\uparrow = \frac{1}{\sqrt{18}}\left\{\left[-2s\uparrow s\uparrow u\downarrow + s\uparrow s\downarrow u\uparrow + s\downarrow s\uparrow u\uparrow\right] + \text{permutations}\right\}$$

$$\Xi^- \uparrow = \frac{1}{\sqrt{18}}\left\{\left[-2s\uparrow s\uparrow d\downarrow + s\uparrow s\downarrow d\uparrow + s\downarrow s\uparrow d\uparrow\right] + \text{permutations}\right\}$$

$$\Lambda^o \uparrow = \frac{1}{\sqrt{12}}\left\{\left[u\uparrow d\downarrow s\uparrow - u\downarrow d\uparrow s\uparrow\right] + \text{permutations}\right\}$$

Note the following for the flavor-spin wave functions of the baryon octet:

1) For p, n, Σ^+, Σ^-, Ξ^o, Ξ^-, they contain two identical quarks and there are three permutations, i.e.

$$\begin{aligned} p\uparrow = \frac{1}{\sqrt{18}}\Big\{&\left[2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow\right] \\ &+\left[2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow\right] \\ &+\left[2d\downarrow u\uparrow u\uparrow - d\uparrow u\uparrow u\downarrow - d\uparrow u\downarrow u\uparrow\right]\Big\} \end{aligned}$$

For Σ^o and Λ, they have three different quarks, and there are six permutations, i.e.

$$\begin{aligned} \Lambda^o\uparrow = \frac{1}{\sqrt{12}}\Big\{&\left[u\uparrow d\downarrow s\uparrow - u\downarrow d\uparrow s\uparrow\right] + \left[d\downarrow u\uparrow s\uparrow - d\uparrow u\downarrow s\uparrow\right] \\ &+\left[s\uparrow d\downarrow u\uparrow - s\uparrow d\uparrow u\downarrow\right] + \left[u\uparrow s\uparrow d\downarrow - u\downarrow s\uparrow d\uparrow\right] \\ &+\left[s\uparrow u\uparrow d\downarrow - s\uparrow u\downarrow d\uparrow\right] + \left[d\downarrow s\uparrow u\uparrow - d\uparrow s\uparrow u\downarrow\right]\Big\} \end{aligned}$$

2) Once one knows the proton wave function, one can obtain the wave functions for other baryons by replacing (interchanging) the quark flavors. The resulting wave function is correct within an overall factor.

$p \leftrightarrow n$ $(u \leftrightarrow d)$	$p \leftrightarrow \Sigma^+$ $(d \leftrightarrow s)$
$\Sigma^+ \leftrightarrow \Sigma^-$ $(u \leftrightarrow d)$	$n \leftrightarrow \Xi^o$ $(d \leftrightarrow s)$
$\Xi^o \leftrightarrow \Xi^-$ $(u \leftrightarrow d)$	

3) The total spin of an identical quark pair is always in an $s = 1$ (symmetric) state. This is required since the overall wave function is anti-symmetric for this quark pair, and the color wave function is anti-symmetric.

4) For Σ^o, the *ud* have $s = 1$, while $s = 0$ for *ud* in Λ^o. Also, for Λ^o the baryon's spin only resides on the strange quark.

Magnetic Dipole Moments of Hadrons

We have obtained the flavor-spin functions of the lightest mesons and baryons. Can this simple constituent quark model describe the static properties of these hadrons, such as mass, spin, parity, magnetic dipole moment, lifetime, decay modes, etc.?

The spin and parity of the mesons and baryons naturally emerge from the quark model. It turns out that the magnetic dipole moments and the masses of hadrons are also well described by the quark model.

First, we consider the magnetic dipole moments.

For point spin-½ charged particles, such as $e^{\pm}$, $\mu^{\pm}$, the Dirac theory tells us that $g = 2$, where g is the gyromagnetic ratio:

$$\vec{\mu} = \frac{Qe}{2mc} g\vec{s}$$

$\vec{\mu}$ is the magnetic dipole moment, m is the mass of the point particle, and $\vec{s}$ is the spin of the particle.

One gets, for example, for electron $\vec{\mu}_e = \frac{-e}{2m_e c} \bullet 2 \bullet \frac{1}{2}\hbar\vec{\sigma} = \frac{-e\hbar}{2m_e c}\vec{\sigma}$

magnitude of $\vec{\mu}_e$ is $\mu_e = \frac{-e\hbar}{2m_e c}$

Similarly, for spin-½ quarks and anti-quarks, the magnetic dipole moments are

$$\mu_u = \frac{{}^2\!/_3\, e\hbar}{2m_u c} = \frac{e\hbar}{3m_u c} \qquad \mu_d = \frac{-{}^1\!/_3\, e\hbar}{2m_d c} = -\frac{e\hbar}{6m_d c}$$

$$\mu_{\bar{d}} = -\mu_d \qquad \mu_{\bar{u}} = -\mu_u$$

In the quark model, the magnetic dipole moment operator for a hadron is expressed as the vector sum of the magnetic dipole moment operator of the constituent quarks

$$\vec{\mu}_H = \sum_{i=1}^{n} \vec{\mu}_i$$

The magnetic dipole moment is defined as the expectation value of the *z*-component of $\vec{\mu}_H$ for a hadron with spin along the *z* direction. The baryon magnetic dipole moment is therefore $\mu_B = \left\langle B\uparrow \left| \sum_{i=1}^{3} \mu_i (\sigma_z)_i \right| B\uparrow \right\rangle$

As an example, for proton, we have

$$\left| P\uparrow \right\rangle = \frac{1}{\sqrt{18}} \left\{ \left[2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow \right] + \text{permutation} \right\}$$

(Note that the *u*-*u* pairs are always in an $s = 1$ state, since *uu* are identical particles and their spin wave function has to be symmetric to produce an overall anti-symmetric wave function after the color wave function is included.)

$$\mu_p = \frac{1}{18} \left[4(\mu_u + \mu_u - \mu_d) + (\mu_u - \mu_u + \mu_d) + (\mu_u - \mu_u + \mu_d) \right] \times 3$$
$$= \frac{1}{6}(8\mu_u - 4\mu_d + 2\mu_d) = \frac{1}{3}(4\mu_u - \mu_d)$$

For neutron's magnetic dipole moment, we simply interchange $u \leftrightarrow d$ and obtain

$$\mu_n = \frac{1}{3}(4\mu_d - \mu_u)$$

An interesting prediction of the quark model can be obtained if one assumes that the up and down quarks have the same mass. Then,

$$\mu_d = -\frac{1}{2}\mu_u$$

and $$\mu_n/\mu_p = \frac{4\mu_d - \mu_u}{4\mu_u - \mu_d} = \frac{-3\mu_u}{\frac{9}{2}\mu_u} = -\frac{2}{3}$$

Experimentally, $\mu_p = 2.793\left(\frac{e\hbar}{2m_p c}\right)$ and $\mu_n = -1.913\left(\frac{e\hbar}{2m_p c}\right)$

$\mu_n/\mu_p = -0.685$, very close to the quark model calculation of $-\frac{2}{3}$.

From the proton's magnetic dipole moment, one can even obtain an estimate of the *u*, *d* quark mass.

$$\mu_p = \frac{1}{3}(4\mu_u - \mu_d) = \frac{3}{2}\mu_u = \frac{3}{2}\left(\frac{\frac{2}{3}e\hbar}{2m_u c}\right) = \frac{e\hbar}{2m_u c}$$

but μ_p is measured to be $2.793\left(\frac{e\hbar}{2m_p c}\right)$.

Therefore $$2.793\left(\frac{e\hbar}{2m_p c}\right) = \frac{e\hbar}{2m_u c} \qquad m_u = \frac{1}{2.793}m_p = 336 \text{ MeV}$$

For the Λ-baryon magnetic dipole moment, we have

$$|\Lambda\uparrow\rangle = \frac{1}{\sqrt{12}}\left\{\left[u\uparrow d\downarrow s\uparrow - u\downarrow d\uparrow s\uparrow\right] + \text{permutation}\right\}$$

(Note that the *u*, *d* pair have spin = 0, and the spin of Λ is carried by the *s* quark alone.)

$$\mu_\Lambda = \frac{1}{12}\left[\mu_u - \mu_d + \mu_s - \mu_u + \mu_d + \mu_s\right] \times 6 = \mu_s$$

From the experiment, $\mu_\Lambda = -0.61\left(\frac{e\hbar}{2m_p c}\right)$

This should be equal to μ_s according to the quark model.

Therefore, $$\mu_\Lambda = -0.61\left(\frac{e\hbar}{2m_p c}\right) = \mu_s \frac{-\left(1/3\right)e\hbar}{2m_s c}$$

and $$m_s = \left(\frac{1}{0.61}\right)\left(\frac{1}{3}\right)m_p = 513 \text{ MeV}$$

One can also predict the magnetic dipole moments for $\Sigma^\pm$, Σ^o, Ξ^-, Ξ^o using the quark model. Reasonably good agreements with the experimental data (within 20%) are obtained.

Masses of Hadrons

We now consider the masses of hadrons in the quark model. For hadrons with zero orbital angular momenta, the interaction between quarks can depend on the relative orientation of their spins. An empirical expression for the masses of mesons and baryons are given as

$$M\ (\text{meson}) = m_1 + m_2 + a\frac{\vec{\sigma}_1 \bullet \vec{\sigma}_2}{m_1 m_2}$$
$$M\ (\text{baryon}) = m_1 + m_2 + m_3 + \frac{a'}{2}\sum_{i>j}\frac{\vec{\sigma}_i \bullet \vec{\sigma}_j}{m_i m_j}$$

The $\vec{\sigma}_i \bullet \vec{\sigma}_j$ term is the QCD analog of the 'hyperfine' interaction in QED, which originates from magnetic force. This term is called color-magnetic interaction. m_1, m_2, m_3 correspond to the masses of the quarks.

It is relatively straight forward to calculate the meson mass. Note that

$$\begin{aligned}\vec{\sigma}_1 \bullet \vec{\sigma}_2 &= 4\left(\vec{s}_1 \bullet \vec{s}_2\right) = 4\left[\left(\vec{s}_1 + \vec{s}_2\right) - \vec{s}_1^{\,2} - \vec{s}_2^{\,2}\right]/2 \\ &= 2\left[s(s+1) - s_1(s_1+1) - s_2(s_2+1)\right] \\ &= 2\left[s(s+1) - \frac{3}{2}\right]\end{aligned}$$

$$= 1 \text{ for } s = 1, \text{ or } -3 \text{ for } s = 0 \text{ where } \vec{s} = \vec{s}_1 + \vec{s}_2$$

Using $m_u = m_d = 310$ MeV, $m_s = 483$ MeV, $a = 160$ MeV (m_u^2) one obtains the following meson masses (in MeV):

	$\pi^\pm$	ρ	K	K^*	η	φ
calculated	140	780	484	890	559	1032
observed	138	776	496	892	549	1020

For the mass of baryon, the quark model calculation is slightly more complicated.

$$M \text{ (baryon)} = m_1 + m_2 + m_3 + \frac{a'}{2}\left(\frac{\vec{\sigma}_1 \bullet \vec{\sigma}_2}{m_1 m_2} + \frac{\vec{\sigma}_2 \bullet \vec{\sigma}_3}{m_2 m_3} + \frac{\vec{\sigma}_1 \bullet \vec{\sigma}_3}{m_1 m_3}\right)$$

a) For baryons made of u, d quarks only (n, p, Δ) or of s quarks only (Ω), $m_1 = m_2 = m_3$ and the color-magnetic term becomes

$$\frac{a'}{2}\frac{1}{m^2}\left(\vec{\sigma}_1 \bullet \vec{\sigma}_2 + \vec{\sigma}_2 \bullet \vec{\sigma}_3 + \vec{\sigma}_1 \bullet \vec{\sigma}_3\right)$$

where $\vec{\sigma}_1 \bullet \vec{\sigma}_2 + \vec{\sigma}_2 \bullet \vec{\sigma}_3 + \vec{\sigma}_1 \bullet \vec{\sigma}_3 = 2\left(\vec{s}^{\,2} - \vec{s}_1^{\,2} - \vec{s}_2^{\,2} - \vec{s}_3^{\,2}\right) = 2\left(s(s+1) - 9/4\right)$

s is the total spin of the baryon $\left(\vec{s} = \vec{s}_1 + \vec{s}_2 + \vec{s}_3\right)$

b) For $J = 3/2$ baryon decuplet

$$\vec{\sigma}_1 + \vec{\sigma}_2 = 2\left(\vec{s}_1 + \vec{s}_2\right) = 2\vec{s}_{12} \qquad s_{12} = 1$$

Similarly for s_{23} and s_{13}

Therefore, $\vec{\sigma}_1 + \vec{\sigma}_2 = 2\left(\vec{s}_{12}^{\,2} - \vec{s}_1^{\,2} - \vec{s}_2^{\,2}\right) = 2\left(2 - 3/2\right) = 1$

Similarly, for $\vec{\sigma}_1 \bullet \vec{\sigma}_3$, $\vec{\sigma}_2 \bullet \vec{\sigma}_3$

c) For $J = ½$ baryon octet

Σ^+ (uus), Σ^- (dds), Ξ^o (ssu), Ξ^- (ssd), Σ^o (uds)

All have $m_1 = m_2 = m$
In addition, the first two quarks couple to $s_{12} = 1$

Therefore $\left(\frac{\vec{\sigma}_1\bullet\vec{\sigma}_2}{m_1 m_2}+\frac{\vec{\sigma}_2\bullet\vec{\sigma}_3}{m_2 m_3}+\frac{\vec{\sigma}_1\bullet\vec{\sigma}_3}{m_1 m_3}\right)=\frac{\vec{\sigma}_1\bullet\vec{\sigma}_2}{m^2}+\frac{\vec{\sigma}_2\bullet\vec{\sigma}_3}{mm_3}+\frac{\vec{\sigma}_1\bullet\vec{\sigma}_3}{mm_3}$

$$=\frac{2\left(\vec{s}_{12}^{\,2}-\vec{s}_1^{\,2}-\vec{s}_2^{\,2}\right)}{m^2}+\frac{1}{mm_3}\left(\vec{\sigma}_1\bullet\vec{\sigma}_2+\vec{\sigma}_2\bullet\vec{\sigma}_3+\vec{\sigma}_1\bullet\vec{\sigma}_3\right)-\frac{\vec{\sigma}_1\bullet\vec{\sigma}_2}{mm_3}$$

$$=2\left[s_{12}\left(s_{12}+1\right)-3/2\right]\left[\frac{1}{m^2}-\frac{1}{mm_3}\right]+\frac{1}{mm_3}\left[2\left(s\left(s+1\right)-9/4\right)\right]$$

d) For Λ^o (uds), it is similar to c), except that $s_{12} = 0$.

Nuclei are made of protons and neutrons. In certain sense, they are analogous to hadrons which are made of quarks and anti-quarks. However, protons and neutrons are color-less objects, unlike the quarks and anti-quarks. This has interesting implications which can be illustrated by a consideration of the mass-3 nuclear system ^{3}H and ^{3}He.

^{3}H consists of (pnn) while ^{3}He is made of (ppn). (p, n) are isospin doublet, just like (d, u) form and isospin doublet. If one substitutes (p, n) by (u, d), then

$$\underset{^3\mathrm{H}}{(pnn)} \rightarrow \underset{n}{(udd)} \quad \text{and} \quad \underset{^3\mathrm{He}}{(ppn)} \rightarrow \underset{p}{(uud)}.$$

In the previous sections, we discussed the wave functions and magnetic moments of p, n in terms of the constituent quarks (u, d). Can one extend such an approach to describe ^{3}H and ^{3}He?

Let us consider the magnetic moments of ^{3}H and ^{3}He. We recall that the magnetic moment of proton is expressed in terms of the magnetic moment of up, down quarks:

$$\mu_p=\frac{1}{3}\left(4\mu_u-\mu_d\right)$$

Similarly, for neutron: $$\mu_n=\frac{1}{3}\left(4\mu_d-\mu_u\right)$$

If the wave functions for ^{3}He, ^{3}H are analogous to those of *p*, *n*, then one could express the magnetic moments of ^{3}He, ^{3}H in terms of the magnetic moments of *p* and *n*

$$\mu_{^3\text{He}} = \frac{1}{3}\left(4\mu_p - \mu_n\right) \qquad \mu_{^3\text{H}} = \frac{1}{3}\left(4\mu_n - \mu_p\right)$$

Since μ_p = 2.79 N.M, μ_n = -1.91 N.M., the above equations give

$$\mu_{^3\text{He}} = 4.36 \text{ N.M.} \qquad \mu_{^3\text{H}} = -3.48 \text{ N.M.}$$

This result is in total disagreement with the experimental data of

$$\mu_{^3\text{He}} = -2.13 \text{ N.M.} \qquad \mu_{^3\text{H}} = 2.98 \text{ N.M.}$$

In fact, even the signs are wrong!

Since proton and neutron are color-neutral objects, the color wavefunctions are symmetric. Therefore, one should calculate the ^{3}H and ^{3}He magnetic moments assuming that their color wave functions are symmetric (and hence the flavor-spin wave functions are anti-symmetric). The expressions for ^{3}H and ^{3}He's magnetic moments are now very different. We obtain

$$\mu_{^3\text{He}} = \mu_n = -1.91 \text{ N.M.} \qquad \mu_{^3\text{H}} = \mu_p = 2.79 \text{ N.M.}$$

These results are much closer to the experimental values of $\mu_{^3\text{He}} = -2.13 \text{ N.M.}$ and $\mu_{^3\text{H}} = 2.98 \text{ N.M.}$

This example illustrates the important role played by color for describing the hadrons. For ^{3}He, ^{3}H, the underlying nucleons are colorless, resulting in very different magnetic moments from what would have been expected assuming a complete analogy between (^{3}He, ^{3}H) and (*p*, *n*).

Another interesting example is the magnetic dipole moment of deuteron. If one assumes a simple wave function for deuteron:

$$d\uparrow = p\uparrow n\uparrow \quad \text{(deuteron has total spin = 1)}$$

This is analogous to vector mesons, whose magnetic moments are given as the sum of the q and $\overline{q}$'s magnetic moments. One expects

$$\left\langle d\uparrow \left|\mu\right| d\uparrow\right\rangle = \mu_p + \mu_n = 2.79 - 1.91 \text{ N.M.} = 0.88 \text{ N.M.}$$

Experimentally, $\mu_d = 0.86$ N.M. The small discrepancy between the simple model prediction and the data reflects a small contribution from the $^3D_1$ state component in the deuteron wave function.

<u>Other Hadrons</u>

We have so far only discussed the lightest mesons and baryons. While these hadrons are among the most important ones to be familiar with, clearly other hadrons can be formed by orbital or radial excitation of one or more quarks, or by replacing the *u*/*d*/*s* quarks by the heavy charm (c) or beauty (b) quarks.

We now consider mesons with total spin $\vec{J}$, orbital angular momentum $\vec{L}$, and quark spin $\vec{S}$

$$\vec{J} = \vec{L} + \vec{S} \qquad\qquad S = 0,\ 1$$

The parity operator P applied to a $q\overline{q}$ meson gives

$$P = \left(-1\right)^L \eta_q \eta_{\overline{q}} = \left(-1\right)^{L+1}$$

where $(-1)^L$ reflects the effect of space inversion on the orbital angular momentum. η_q and $\eta_{\overline{q}}$ are the intrinsic parity of q and $\overline{q}$, respectively. The opposite intrinsic parity of fermion and anti-fermion introduces a factor of -1.

The operation of charge conjugation changes a particle into an anti-particle. For neutral mesons which are their own anti-particles, they are eigenstates of the charge conjugation operator C. The C operation may be considered equivalent to the exchange of q with $\overline{q}$. On exchange, a factor of $(-1)^L$ comes from the orbital angular momentum, another factor of (-1) from the opposite intrinsic parity of q and $\overline{q}$, and finally a factor of $(-1)^{S+1}$ comes from the symmetry property of the spin wave function.

The C-parity for a self-conjugate meson is

$$C = (-1)\,(-1)^{L}\,(-1)^{S+1} = (-1)^{L+S}$$

Since $\qquad P = (-1)^{L+1} \qquad C = (-1)^{L+S} \qquad PC = (-1)^{S+1}$

Hence, for $S = 0$ mesons $PC = -1$ (P, C have opposite signs)
for $S = 1$ mesons, $PC = +1$ (P, C have same signs)

The J^{PC} of ordinary $q\bar{q}$ mesons are as follows:

	$S = 0$	$S = 1$
$L = 0$	0^{-+}	1^{--}
$L = 1$	1^{+-}	$0^{++}, 1^{++}, 2^{++}$
$L = 2$	2^{-+}	$1^{--}, 2^{--}, 3^{--}$
$L = 3$	3^{+-}	$2^{++}, 3^{++}, 4^{++}$
.	.	.
.	.	.
.	.	.

It is important to note that ordinary $q\bar{q}$ mesons cannot have

$$J^{PC} = 0^{--}, 0^{+-}, 1^{-+}, 2^{+-}, 3^{-+}, \ldots\ldots$$

These are 'exotic' quantum numbers, which could only be associated with non-$q\bar{q}$ exotic mesons such as gg (glueball), $q\bar{q}q$ (hybrid), $q^2\bar{q}^2$, etc.

Note that K^{o}, $\bar{K}^{o}$ are neutral mesons, but not eigenstates of C-parity.

Photon is an eigenstate of C. Since EM field changes sign under C, photon has $C = -1$.

C is conserved in strong and electromagnetic interactions.

$$\underset{C=+1}{\pi^{o}} \rightarrow \gamma\gamma \quad \text{is allowed, while} \quad \pi^{o} \not\rightarrow \gamma\gamma\gamma$$

Similarly, $\qquad \underset{C=-1}{\rho^{o}} \not\rightarrow \pi^{o}\pi^{o} \qquad\qquad \rho^{o} \not\rightarrow \pi^{o}\eta$

The current naming scheme for mesons considers two separate cases:

a) "Neutral-flavor" mesons ($S = C = B = 0$)

$q\bar{q}$ content	${}^{2S+1}L_J = {}^1(L_{\text{even}})_J$	${}^1(L_{\text{odd}})_J$	${}^3(L_{\text{even}})_J$	${}^3(L_{\text{odd}})_J$
$u\bar{d}$, $u\bar{u} - d\bar{d}$, $d\bar{u}$ ($I = 1$)	π	b	ρ	a
$d\bar{d} + u\bar{u}$ and/or $s\bar{s}$ ($I = 0$)	η, η'	h, h'	ω, φ	f, f'
$c\bar{c}$	η_c	h_c	ψ	χ_c
$b\bar{b}$	η_b	h_b	Y	χ_b

b) Mesons with non-zero S, C, B

strange meson: K (I = ½)
charm meson: D_S (I = 0), D (I = ½)
beauty meson: B_S (I = 0), B_C (I = 0), B (I = ½)

K, D, B without subscript indicates that they also contain a light u/d or $\bar{u}/\bar{d}$.

The following table from the most recent PDG lists the quark-model assignments for the known mesons:

Table 13.2: Suggested $q\bar{q}$ quark-model assignments for most of the known mesons. Some assignments, especially for the 0^{++} multiplet and for some of the higher multiplets, are controversial. Mesons in bold face are included in the Meson Summary Table. Of the light mesons in the Summary Table, the $f_0(1500)$, $f_1(1510)$, $f_2(1950)$, $f_2(2300)$, $f_2(2340)$, and one of the two peaks in the $\eta(1440)$ entry are not in this table. Within the $q\bar{q}$ model, it is especially hard to find a place for the first two of these f mesons and for one of the $\eta(1440)$ peaks. See the "Note on Non-$q\bar{q}$ Mesons" at the end of the Meson Listings.

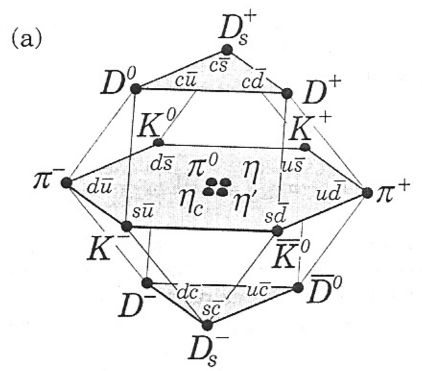
$N\,^{2S+1}L_J$	J^{PC}	$u\bar{d}$, $u\bar{u}$, $d\bar{d}$ $I=1$	$u\bar{u}$, $d\bar{d}$, $s\bar{s}$ $I=0$	$c\bar{c}$ $I=0$	$b\bar{b}$ $I=0$	$\bar{s}u$, $\bar{s}d$ $I=1/2$	$c\bar{u}$, $c\bar{d}$ $I=1/2$	$c\bar{s}$ $I=0$	$\bar{b}u$, $\bar{b}d$ $I=1/2$	$\bar{b}s$ $I=0$	$\bar{b}c$ $I=0$
$1\,^1S_0$	0^{-+}	π	η, η'	$\eta_c(1S)$	$\eta_b(1S)$	K	D	D_s	B	B_s	B_c
$1\,^3S_1$	1^{--}	ρ	ω, ϕ	$J/\psi(1S)$	$\Upsilon(1S)$	$K^*(892)$	$D^*(2010)$	D_s^*	B^*	B_s^*	
$1\,^1P_1$	1^{+-}	$b_1(1235)$	$h_1(1170)$, $h_1(1380)$	$h_c(1P)$		K_{1B}†	$D_1(2420)$	$D_{s1}(2536)$			
$1\,^3P_0$	0^{++}	$a_0(1450)^*$	$f_0(1370)^*$, $f_0(1710)^*$	$\chi_{c0}(1P)$	$\chi_{b0}(1P)$	$K_0^*(1430)$					
$1\,^3P_1$	1^{++}	$a_1(1260)$	$f_1(1285)$, $f_1(1420)$	$\chi_{c1}(1P)$	$\chi_{b1}(1P)$	K_{1A}†					
$1\,^3P_2$	2^{++}	$a_2(1320)$	$f_2(1270)$, $f_2'(1525)$	$\chi_{c2}(1P)$	$\chi_{b2}(1P)$	$K_2^*(1430)$	$D_2^*(2460)$				
$1\,^1D_2$	2^{-+}	$\pi_2(1670)$	$\eta_2(1645)$, $\eta_2(1870)$			$K_2(1770)$					
$1\,^3D_1$	1^{--}	$\rho(1700)$	$\omega(1650)$	$\psi(3770)$		$K^*(1680)$‡					
$1\,^3D_2$	2^{--}					$K_2(1820)$					
$1\,^3D_3$	3^{--}	$\rho_3(1690)$	$\omega_3(1670)$, $\phi_3(1850)$			$K_3^*(1780)$					
$1\,^3F_4$	4^{++}	$a_4(2040)$	$f_4(2050)$, $f_4(2220)$			$K_4^*(2045)$					
$2\,^1S_0$	0^{-+}	$\pi(1300)$	$\eta(1295)$, $\eta(1440)$	$\eta_c(2S)$		$K(1460)$					
$2\,^3S_1$	1^{--}	$\rho(1450)$	$\omega(1420)$, $\phi(1680)$	$\psi(2S)$	$\Upsilon(2S)$	$K^*(1410)$‡					
$2\,^3P_2$	2^{++}	$a_2(1700)$	$f_2(1950)$, $f_2(2010)$		$\chi_{b2}(2P)$	$K_2^*(1980)$					
$3\,^1S_0$	0^{-+}	$\pi(1800)$	$\eta(1760)$			$K(1830)$					

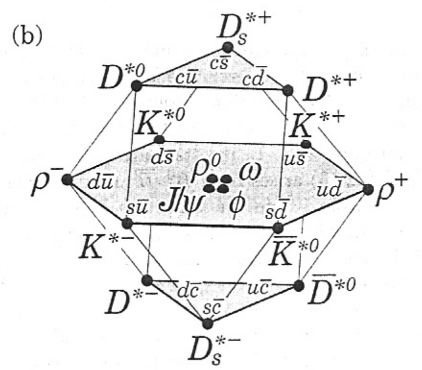
* See our scalar minireview in the Particle Listings. The candidates for the $I=1$ states are $a_0(980)$ and $a_0(1450)$, while for $I=0$ they are: $f_0(600)$, $f_0(980)$, $f_0(1370)$, and $f_0(1710)$. The light scalars are problematic, since there may be two poles for one $q\bar{q}$ state and $a_0(980)$, $f_0(980)$ may be $K\bar{K}$ bound states.

† The K_{1A} and K_{1B} are nearly equal (45°) mixes of the $K_1(1270)$ and $K_1(1400)$.

‡ The $K^*(1410)$ could be replaced by the $K^*(1680)$ as the $2\,^3S_1$ state.

The SU(4) 16-plets for pseudoscalar and vector mesons are shown as follows:





Note that the neutral mesons at the center of the center plane are mixtures of $u\bar{u}$, $d\bar{d}$, $s\bar{s}$ and $c\bar{c}$ states. Also note the 4-fold symmetry in the SU(4) 16-plets. For example, one can form an octet plane containing D^o, D_S^+, K^-, K^+, D_S^-, $\bar{D}^o$ (an SU(3) octet made of u, s, c). Similarly octet planes made of (d, s, c) and (u, d, c) can also be envisaged.

Analogous SU(4) 16-plets can be constructed for (u, d, s, b) and for any orbitally or radially excited mesonic states.

Baryon Excited States

For baryons with $L = 0$, the space wave function is symmetric and the flavor-spin wave function is totally symmetric too. For baryons with one quark excited to the $1P$ state, both symmetric and mixed symmetry space wave function can be formed.

Recall the SU(3), SU(2) (flavor, spin) multiplets:

Symmetric: (10, 4) + (8, 2) = 56

Mixed Symmetry: (10, 2) + (8, 4) +(8, 2) + (1, 2) = 70

Anti-symmetric: (8, 2) + (1, 4) = 20

For baryon 'ground' states with $L = 0$:	(10, 4)	:	$J^P = 3/2^+$
	(8, 2)	:	$J^P = 1/2^+$

For baryon states with one quark excited to $L = 1$, the space-symmetric state vanishes if the origin is chosen to be the center-of-mass of the three quark system (see F. Close's book, page 82). Therefore, only mixed symmetry states are allowed:

$L = 1$: $(L = 1) \otimes (10, 2) \Rightarrow J^P = 3/2^-, 1/2^-$
$(L = 1) \otimes (8, 4) \Rightarrow J^P = 5/2^-, 3/2^-, 1/2^-$
$(L = 1) \otimes (8, 2) \Rightarrow J^P = 3/2^-, 1/2^-$
$(L = 1) \otimes (1, 2) \Rightarrow J^P = 3/2^-, 1/2^-$

The following table lists the $\left(D,\ L_N^P\right)$ for many of the known (u, d, s) baryons. D is the dimensionality of the SU(3) x SU(2), L is the total quark orbital angular momentum, and P is the total parity. N is the total number of quanta of excitation (both radial and orbital). Note that baryon states with total spin up to $11/2^+$ have been observed.

Table 13.4: Quark-model assignments for many of the known baryons in terms of a flavor-spin SU(6) basis. Only the dominant representation is listed. Assignments for some states, especially for the $\Lambda(1810)$, $\Lambda(2350)$, $\Xi(1820)$, and $\Xi(2030)$, are merely educated guesses. For assignments of the charmed baryons, see the "Note on Charmed Baryons" in the Particle Listings.

J^P	(D, L_N^P)	S	Octet members				Singlets
$1/2^+$	$(56,0_0^+)$	1/2	$N(939)$	$\Lambda(1116)$	$\Sigma(1193)$	$\Xi(1318)$	
$1/2^+$	$(56,0_2^+)$	1/2	$N(1440)$	$\Lambda(1600)$	$\Sigma(1660)$	$\Xi(?)$	
$1/2^-$	$(70,1_1^-)$	1/2	$N(1535)$	$\Lambda(1670)$	$\Sigma(1620)$	$\Xi(?)$	$\Lambda(1405)$
$3/2^-$	$(70,1_1^-)$	1/2	$N(1520)$	$\Lambda(1690)$	$\Sigma(1670)$	$\Xi(1820)$	$\Lambda(1520)$
$1/2^-$	$(70,1_1^-)$	3/2	$N(1650)$	$\Lambda(1800)$	$\Sigma(1750)$	$\Xi(?)$	
$3/2^-$	$(70,1_1^-)$	3/2	$N(1700)$	$\Lambda(?)$	$\Sigma(?)$	$\Xi(?)$	
$5/2^-$	$(70,1_1^-)$	3/2	$N(1675)$	$\Lambda(1830)$	$\Sigma(1775)$	$\Xi(?)$	
$1/2^+$	$(70,0_2^+)$	1/2	$N(1710)$	$\Lambda(1810)$	$\Sigma(1880)$	$\Xi(?)$	$\Lambda(?)$
$3/2^+$	$(56,2_2^+)$	1/2	$N(1720)$	$\Lambda(1890)$	$\Sigma(?)$	$\Xi(?)$	
$5/2^+$	$(56,2_2^+)$	1/2	$N(1680)$	$\Lambda(1820)$	$\Sigma(1915)$	$\Xi(2030)$	
$7/2^-$	$(70,3_3^-)$	1/2	$N(2190)$	$\Lambda(?)$	$\Sigma(?)$	$\Xi(?)$	$\Lambda(2100)$
$9/2^-$	$(70,3_3^-)$	3/2	$N(2250)$	$\Lambda(?)$	$\Sigma(?)$	$\Xi(?)$	
$9/2^+$	$(56,4_4^+)$	1/2	$N(2220)$	$\Lambda(2350)$	$\Sigma(?)$	$\Xi(?)$	
			Decuplet members				
$3/2^+$	$(56,0_0^+)$	3/2	$\Delta(1232)$	$\Sigma(1385)$	$\Xi(1530)$	$\Omega(1672)$	
$1/2^-$	$(70,1_1^-)$	1/2	$\Delta(1620)$	$\Sigma(?)$	$\Xi(?)$	$\Omega(?)$	
$3/2^-$	$(70,1_1^-)$	1/2	$\Delta(1700)$	$\Sigma(?)$	$\Xi(?)$	$\Omega(?)$	
$5/2^+$	$(56,2_2^+)$	3/2	$\Delta(1905)$	$\Sigma(?)$	$\Xi(?)$	$\Omega(?)$	
$7/2^+$	$(56,2_2^+)$	3/2	$\Delta(1950)$	$\Sigma(2030)$	$\Xi(?)$	$\Omega(?)$	
$11/2^+$	$(56,4_4^+)$	3/2	$\Delta(2420)$	$\Sigma(?)$	$\Xi(?)$	$\Omega(?)$	

Baryon Naming Scheme

The naming scheme for baryons containing *u*, *d*, *s*, *c*, *b* quarks is relatively straight-forward. The symbols N, Δ, Λ, Σ, Ξ, and Ω originally used for the baryons made of light quarks (*u*, *d*, *s*) are now generalized for heavy baryons too.

3 *u*/*d* : N (Isospin = ½), Δ ($I = 3/2$)

2 *u*/*d* : Λ ($I = 0$), Σ ($I = 1$)

If the third quark is a *c* or *b* quark, its identity is given by a subscript, i.e.
Λ_b, Λ_c, Σ_b, Σ_c

1 *u*/*d* : Ξ. They have $I = ½$.

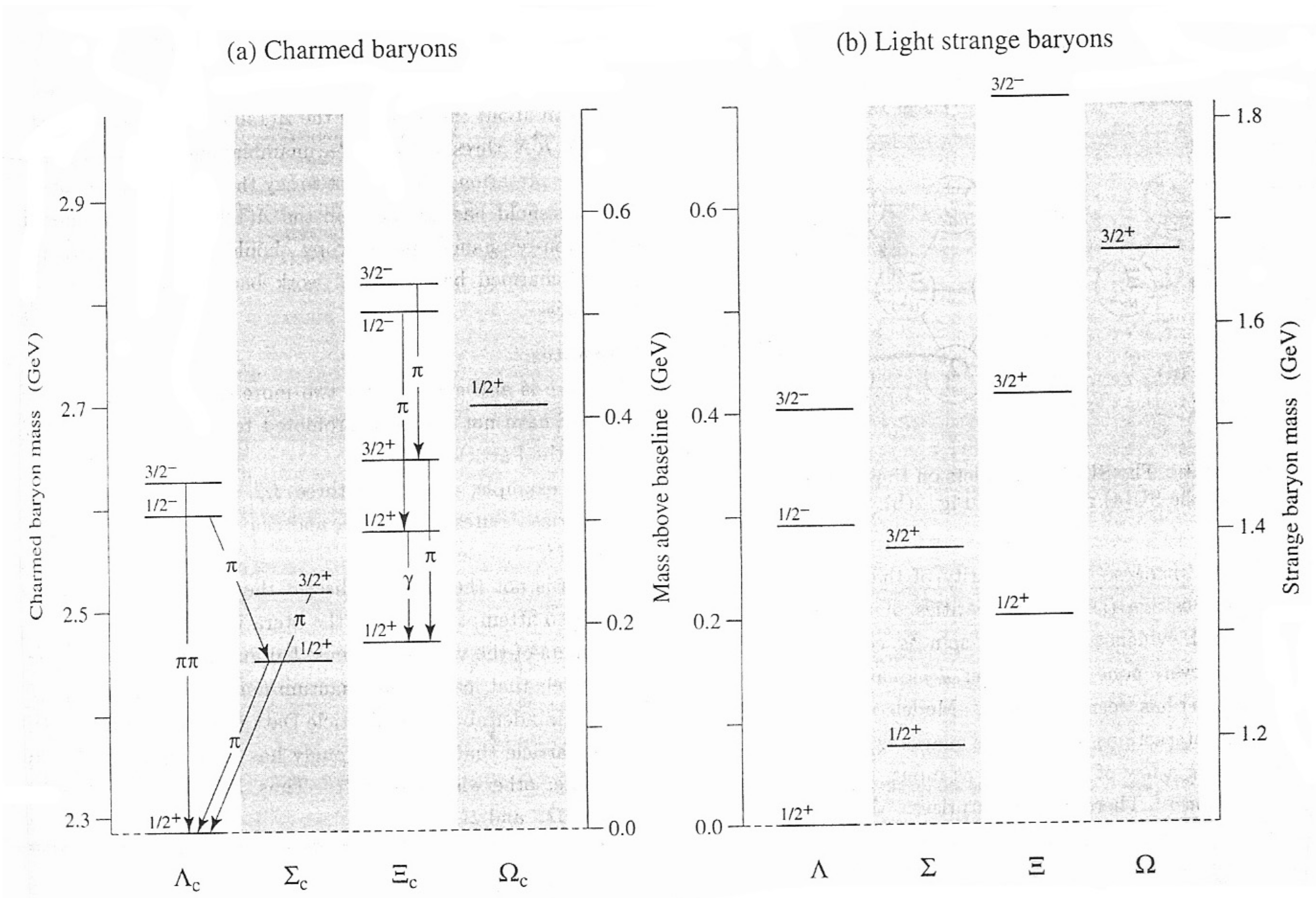
If one or both of the non-*u*/*d* quarks are heavy, then subscripts are added:
Ξ_c, Ξ_{cc}, Ξ_b, Ξ_{bc}, Ξ_{bs}, etc (Note that Ξ_c contains (*u*/*d*) *s c* where *s* is not specified in the subscript.

No *u*/*d* : Ω. They all have $I = 0$.

The heavy quark contents are again shown explicitly in the subscripts, such as Ω_{ccc}, Ω_{cc}, Ω_c, Ω_{bbb}, Ω_{bbc}, Ω_{bcc}, etc.

A baryon that decays strongly has its mass as part of its name, such as $\Delta(1232)$, $\Sigma(1385)$, etc.

There are eleven known charm baryons, each with one c quark. The charmed-baryon spectrum is shown here in comparison with the light baryons:



For *b*-baryons, both Λ_b and Ξ_b have been found.

Charmed Baryons

For baryons containing *u*, *d*, *s*, *c* quarks, the SU(3) needs to be extended to SU(4):

$$4 \otimes 4 \otimes 4 = 4 \oplus 20 \oplus 20 \oplus 20$$

The SU(3) symmetric decuplet is now extended to SU(4) totally symmetric 20-plet. The SU(3) mixed symmetry octet now becomes SU(4) mixed-symmetry 20-plet. These two SU(4) 20-plets are shown as follows:

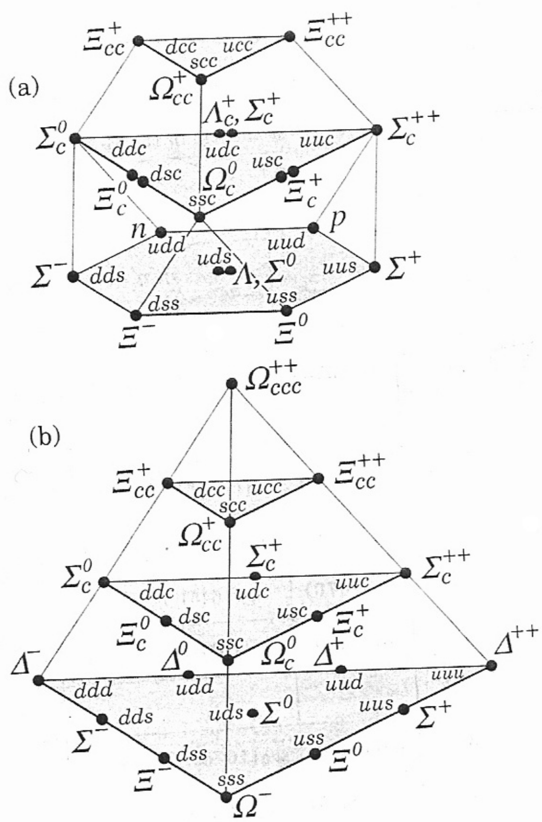


Figure 13.2: SU(4) multiplets of baryons made of u, d, s, and c quarks. (a) The 20-plet with an SU(3) octet. (b) The 20-plet with an SU(3) decuplet.

(a) mixed-symmetry 20-plets (b) symmetric 20-plets

Note the 4-fold symmetry in these diagrams. The mixed-symmetry 20-plets contain 4 SU(3) octets made of (u, d, s), (u, d, c), (u, s, c) and (d, s, c). Similarly, the symmetric 20-plets contain 4 SU(3) decuplets.

Exotic Mesons and Baryons

While $q\overline{q}$ and qqq states are the simplest colorless hadrons one can construct, there are clearly other colorless states which can be made out of q, $\overline{q}$, and gluons. Extensive experimental searches for such exotic hadrons have not yet provided any clear evidence for their existence. Some examples of the exotic hadrons are:

a) Glueballs. These are colorless objects made of a pair of gluons. Lattice gauge calculations suggest that light glueballs have $J^{PC} = 0^{++}$ and 2^{++} (~ 1600 MeV for 0^{++}

and ~ 2230 for 2^{++}). So far, the best candidate for glueball is the f_o (1500) 0^{++} state. These glueballs can mix with ordinary $q\bar{q}$ states, making their identification a difficult task.

b) Hybrids. These are exotic mesonic states consisting of $q\bar{q}g$, where a gluon combines with the $q\bar{q}$. The presence of the extra gluon allows some exotic quantum numbers for the hybrids. Evidence for such an exotic hybrid with J^{PC} = 1^{-+} has been reported for the $\pi_1(1400)$ produced in the $\pi^- p \rightarrow \eta\pi^- p$ reaction, where $\pi_1(1400)$ couples to $\eta\pi^-$. More extensive studies for the hybrids are planned in the future at the Jefferson Laboratory.

c) Dibaryons. These are six-quark states. The most celebrated example is the 'H particle' made of (*uuddss*). Extensive search has shown no evidence for such particle.