Chapter 2

Symmetry and Conservation Laws

Symmetry and conservation laws are closely connected. Conservation laws are results of symmetries in the physical system. One can separate various symmetries into two categories:

a) Space-time symmetries including space translation, time translation, rotation, Lorentz transformation, space inversion, time-reversal, etc.

b) Other symmetries not related to space-time, such as isospin, permutation symmetry, charge-conjugation, gauge invariance, etc. These can be considered as 'internal' symmetries.

We first consider examples of the space-time symmetries. Suppose we describe a physical system with two different frames of reference S and S'. S and S' are related by

$$(t,\, x,\, y,\, z) \rightarrow (t',\, x',\, y',\, z')$$

The transformation can be specified by the number of parameters it contains. For example, a time translation $t \rightarrow t' = t + \tau$ is specified by a single parameter τ. Space translation, $\vec{v}' = \vec{v} + \vec{a}$, is specified by three parameters (a_x, a_y, a_z). Rotation (reorientation) of the coordinate system is also represented by three parameters.

In general, these transformations form families or groups of transformation, and they have properties of a group. We recall the properties of a group:

1) A law of combination, usually called multiplication, is defined such that

$$\text{If } a \in G,\ b \in G, \text{ then } ab \in G$$

2) There exists an identity element $e \in G$. For all a in G, $ea = ae = a$

3) For every $a \in G$, there exists $a^{-1} \in G$ such that

$$a^{-1}a = aa^{-1} = e$$

4) The law of combination is associative

$$(ab)c = a(bc)$$

The simplest group consists of one element e with the multiplication law $ee = e$.

The next simplest group consists of two elements e, a with a law of multiplication

$$ee = e,\ ea = ae = a,\ aa = e$$

These groups are called Abelian groups, since all group elements commute.

The smallest non-Abelian group is the group of permutation of three objects:

$$e = (1\ 2\ 3) \quad a = (2\ 1\ 3) \quad b = (1\ 3\ 2)$$
$$c = (3\ 2\ 1) \quad d = (3\ 1\ 2) \quad e = (2\ 3\ 1)$$

note: $ab = (3\ 1\ 2)$ while $ba = (2\ 3\ 1)$

$$ab \neq ba$$

The permutation group of n objects is called the symmetric group of degree n: S_n

Note that S_2 consists of two elements: $e = (1\ 2)$, $a = (2\ 1)$,
and it has the multiplication law $e^2 = e$, $a^2 = e$, $ae = ea = a$, just like the simple Abelian group of two elements mentioned earlier.

Now consider the space translation

$$\vec{r}' = \vec{r} + \vec{a},\ \vec{r}'' = \vec{r}' + \vec{b}$$

a) Sequential $\vec{r} \to \vec{r}' \to \vec{r}''$ transformation gives $\vec{r}'' = \vec{r} + \left(\vec{a} + \vec{b}\right)$, which is also a space translation.

b) $\vec{a} = 0$ is the identity transformation.

c) $\left(\vec{a}\right)^{-1} = -\vec{a}$ is clearly the inverse transformation.

d) Finally, $\vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c}$. Hence, the transformation is associative.

a) through d) show that space translation forms a group.

Lorentz transformation also forms a group.

Recall that Lorentz transformation is given as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

where

$$x^{\nu} = \left(x^0,\ x^1,\ x^2,\ x^3\right) = \left(ct,\ x,\ y,\ z\right)$$

Lorentz transformation leaves x^2 invariant:

$$x^2 = x^{\mu} x_{\mu} = c^2 t^2 - x^2 - y^2 - z^2$$

$$x_{\mu} = g_{\mu\nu} x^{\nu} \qquad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x'^2 = x'^{\mu} x'_{\mu} = x'^{\mu} g_{\mu\nu} x'^{\nu} = \Lambda^{\mu}{}_{\alpha} x^{\alpha} g_{\mu\nu} \Lambda^{\nu}{}_{\beta} x^{\beta}$$

but, $$x^2 = x^{\alpha} g_{\alpha\beta} x^{\beta}$$

and Lorentz transformation leaves x^2 invariant:

$$x'^2 = x^2 \Rightarrow \Lambda^{\mu}{}_{\alpha} g_{\mu\nu} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$$

or

$$\left(\Lambda^T\right)_{\alpha}{}^{\mu} g_{\mu\nu} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$$

Hence $\Lambda^T g \Lambda = g$ defines Lorentz transformation

To show that Lorentz transformations form a group:

a) Let $\Lambda_1^T g \Lambda_1 = g \qquad \Lambda_2^T g \Lambda_2 = g$

then $\Lambda_3 = \Lambda_2 \Lambda_1$

is also a Lorentz transformation.

$$\Lambda_3^T g \Lambda_3 = \left(\Lambda_2 \Lambda_1\right)^T g \left(\Lambda_2 \Lambda_1\right)$$
$$= \Lambda_1^T \Lambda_2^T g \Lambda_2 \Lambda_1 = \Lambda_1^T g \Lambda_1 = g$$
$$\Lambda_3^T g \Lambda_3 = g$$

showing that, if Λ_1, Λ_2 are Lorentz transformations, the $\Lambda_3 = \Lambda_2 \Lambda_1$ is also a Lorentz transformation.

b) Identity element is clearly $\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

c) First we show that for any Λ_1, there exists an inverse Λ_1^{-1}.

$$\Lambda_1^T g \Lambda_1 = g \qquad \det\left(\Lambda_1^T g \Lambda_1\right) = \det(g)$$
$$\det\left(\Lambda_1^T g \Lambda_1\right) = \det\left(\Lambda_1^T\right) \det(g) \det\left(\Lambda_1\right)$$
$$= \left(\det\left(\Lambda_1\right)\right)^2 \det(g) = \det(g)$$

Hence, $\det \Lambda_1 = \pm 1$

and Λ_1 has an inverse Λ_1^{-1}.

We also need to show $\left(\Lambda_1^{-1}\right)^T g \left(\Lambda_1^{-1}\right) = g$

$$\left(\Lambda_1\right)^T g \Lambda_1 = g \Rightarrow \Lambda_1^T g \Lambda_1 \Lambda_1^{-1} = g \Lambda_1^{-1}$$
$$\Rightarrow \left(\Lambda_1^{-1}\right)^T \Lambda_1^T g = \left(\Lambda_1^{-1}\right)^T g \Lambda_1^{-1}$$
$$\Rightarrow \left(\Lambda_1 \Lambda_1^{-1}\right)^T g = g = \left(\Lambda_1^{-1}\right)^T g \Lambda_1^{-1}$$

Therefore $$\left(\Lambda_1^{-1}\right)^T g\Lambda_1^{-1} = g$$

d) $$\left(\Lambda_1\Lambda_2\right)\Lambda_3 = \Lambda_1\left(\Lambda_2\Lambda_3\right)$$

as a result of matrix multiplication definition.

If S and S' are equally valid frames of reference for formulating the laws describing the behavior of a system, then S and S' are related by a ‘symmetry transformation’. In S', the wave function described in system S as $\psi(x)$ now becomes $\psi'(x')$. $\psi'(x')$ has the same value as $\psi(x)$.

$$\psi'(x') = \psi(x) = \psi\left(f^{-1}(x')\right)$$

where $x' = f(x)$ signifies the coordinate transformation.

The above equation can be re-expressed as a function of x, namely

$$\psi'(x) = \psi\left(f^{-1}(x)\right) = U\psi(x)$$

where U is an operator which depends on the coordinate transformation ($x' = f(x)$).

The requirement that the norm $\langle\psi|\psi\rangle$ is preserved in the transformation implies

$$\langle\psi'|\psi'\rangle = \langle U\psi|U\psi\rangle = \langle\psi|U^+U|\psi\rangle$$

Hence, $U^+U = 1$, $U^+ = U^{-1}$ and U is a unitary operator. However, U is not a Hermitian operator and is not necessarily an observable.

However, if the transformation can be built up from an infinitesimal transformation

$$U = 1 + i\delta\alpha G$$

where $\delta\alpha$ is an infinitesimal real number. Then

$$U^+U = 1 \Rightarrow G - G^+ = 0 \qquad G = G^+$$

G is Hermitian, and an observable.

For a symmetry transformation which leaves the Hamiltonian invariant

$$\langle\psi'|H'|\psi'\rangle = \langle\psi'|H|\psi'\rangle = \langle\psi|U^{+}HU|\psi\rangle = \langle\psi|H|\psi\rangle$$

and

$$U^{+}HU = H \qquad U^{-1}HU = H$$
$$[H, U] = 0$$

U is a constant of motion, and similarly for the generator G. Therefore, G is a conserved observable.

Note that for each parameter α, one can obtain the corresponding generator G_α. Therefore, for a transformation involving n parameters, there are n corresponding generators and conserved observables.

For discrete symmetry such as parity, there is no infinitesimal transformation, and hence no corresponding conserved generator. However, the unitary operator for parity satisfies

$$U_p^2 = 1$$

which follows from the fact that if the parity operation is applied twice, one obtains the identity transformation. Since U_p is also unitary

$$U_p^{+}U_p = 1$$

it follows that
$$U_p = U_p^{+}$$

and in this case, the operator U_p is itself Hermitian and can be interpreted as an observable.

As an example, we consider time translation

$$t \longrightarrow t' = t + \tau$$

Time translation invariance implies that the description of a system is independent of the choice of zero from which time is measured.

$$\psi' = U_\tau \psi(t) = \psi\left(f^{-1}(t)\right) = \psi(t-\tau)$$

$$U_{\delta\tau} = 1 + i(\delta\tau)G$$

$$U_{\delta\tau}\psi(t) = \psi(t-\delta\tau) = \psi(t) - \delta\tau\frac{\partial\psi}{\partial t}$$

but $$U_{\delta\tau}\psi(t) = \left(1+i(\delta\tau)G\right)\psi(t) = \psi(t) + i(\delta\tau)G\psi(t)$$

Hence $$iG\psi = -\frac{\partial\psi}{\partial t}$$

$$G = i\frac{\partial}{\partial t} = H$$

H, the Hamiltonian, is the conserved generator for time translation.

Similarly, one can show that $\vec{P} = i\vec{\nabla}$ is the conserved operator for space translation (3 conserved generators – P_x, P_y, P_z – for a 3-parameter transformation).

Finally, $\vec{J}$ is the conserved operator for space rotation with three conserved generators, J_x, J_y, J_z.

We now discuss isospin conservation as an important example of unitary symmetry.

Isospin Conservation in Strong Interaction

In 1932, right after the discovery of the neutron, Heisenberg suggested that the similarity between proton and neutron mass implies that proton and neutron correspond to two degenerate states of strong interaction:

$$H\left|\psi_p\right\rangle = E\left|\psi_p\right\rangle$$
$$H\left|\psi_n\right\rangle = E\left|\psi_n\right\rangle$$

(note that mass (*n*) = 939.56 MeV and mass (*p*) = 938.27 MeV)

This degeneracy reminds us of the two-fold degeneracy of the spin-½ system for $s_z = +½$, $s_z = -½$, and we describe proton and neutron as two different states of the "nucleon", one with 'isospin' up and the other with isospin down.

$$|p\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$
$$|n\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

A 'nucleon' state can be expressed as

$$\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix} = \psi_p\begin{pmatrix}1\\0\end{pmatrix} + \psi_n\begin{pmatrix}0\\1\end{pmatrix}$$

and the transformation

$$\psi_p \to \psi'_p = \alpha\psi_p + \beta\psi_n$$
$$\psi_n \to \psi'_n = \gamma\psi_p + \delta\psi_n$$

would represent an equivalent state as far as strong interaction is concerned.

$$\begin{pmatrix}\psi'_p\\ \psi'_n\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix} = U\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix}$$

The matrix U must be unitary to preserve the norm. Also, this implies $|\det\ U|^2 = 1$.

Choosing det U = +1, the 2 x 2 unitary matrices form the SU(2) group. There are $2^2 - 1 = 3$ independent parameters.

For an infinitesimal transformation $U = 1 + i\xi$, where ξ is a 2 x 2 matrix whose elements are all small quantities. det U = 1 now implies Tr $\xi = 0$ and the condition that U be unitary, $(1 + i\xi)(1-i\xi^+) = 1$, implies $\xi = \xi^+$.

Thus ξ is a 2 x 2 traceless Hermitian matrix. It can be expressed as

$$\xi = \vec{\varepsilon} \bullet \vec{\tau}/2 \qquad \vec{\varepsilon} = (\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3)$$

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\tau_i/2 , \tau_j/2 \right] = i\varepsilon_{ijk} \left(\tau_k/2 \right)$$

A finite transformation can be built from the infinitesimal transformation repeatedly, and one obtains

$$U = \exp(i\vec{\alpha} \bullet \vec{\tau}/2)$$

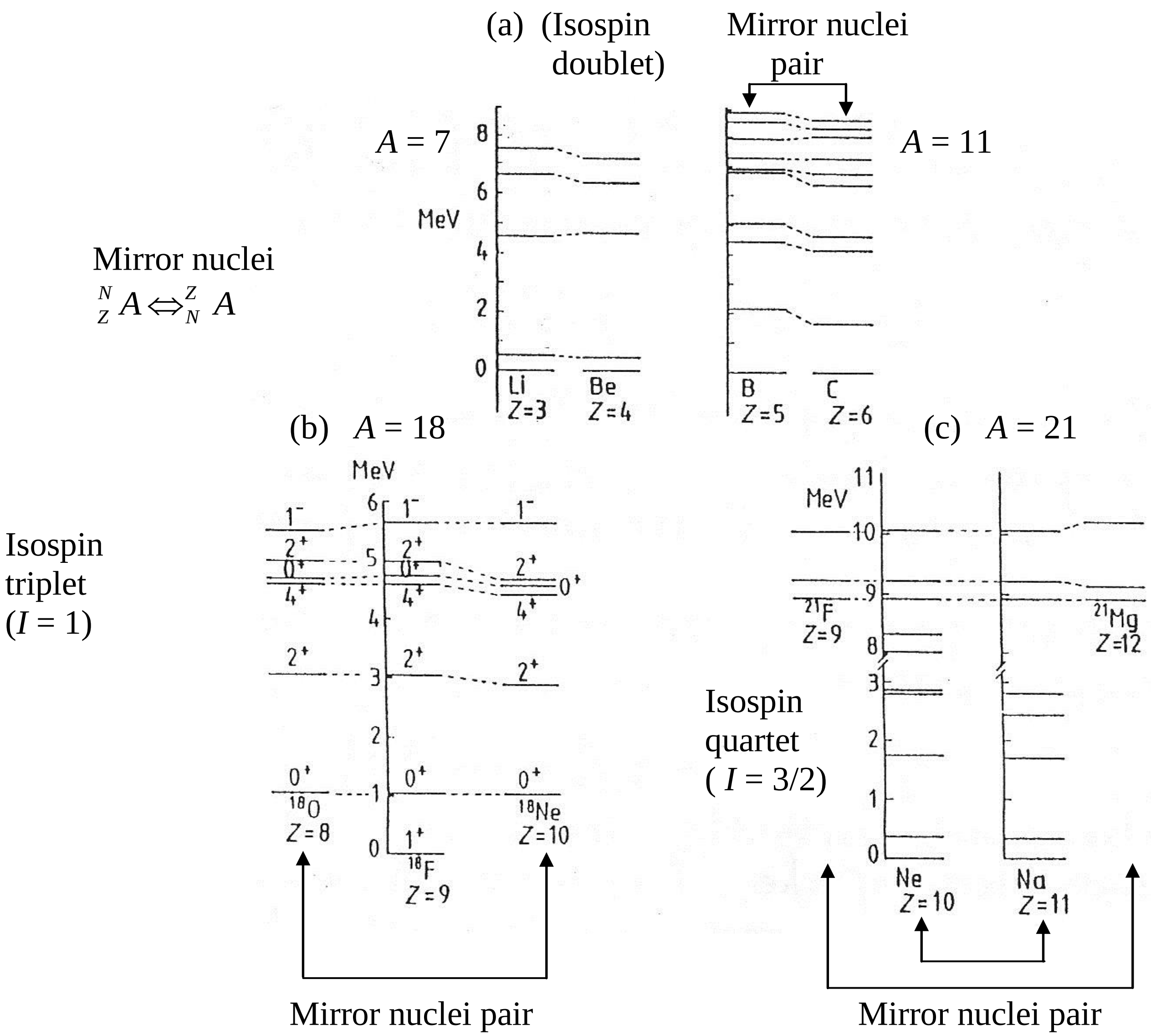
The invariance of the Hamiltonian H in SU(2) transformation leads to $[H, U] = 0$ and $[H, \vec{\tau}\,] = 0$. The eigenvalues of $\vec{\tau}$ are constants of the motion.

Consider states of several nucleons, the total isospin operators

$$\vec{T} = \frac{1}{2}\vec{\tau}_1 + \frac{1}{2}\vec{\tau}_2 + \ldots\ldots \frac{1}{2}\vec{\tau}_A$$

also commute with the Hamiltonian. Thus the eigenvalues of the $\vec{T}$ operators are constant of the motion. Energy levels of nuclei should be characterized by eigenvalues of T^2 and T_3. Isospin should be a good quantum number. For each T, there are $2T + 1$ ‘degenerate’ states.

There are abundant examples for isospin conservation in nuclear physics. As shown in the figure below, the pairs of ‘mirror nuclei’ ${}^7_3\mathrm{Li}/{}^7_4\mathrm{Be}$ and ${}^{11}_5\mathrm{B}/{}^{11}_6\mathrm{C}$ have very similar binding energies for ground state as well as the excited states, after differences in Coulomb interaction (which does not conserve isospin) are corrected for. Similar isospin triplets and isospin quartets are observed for ${}^{18}_8\mathrm{O}/{}^{18}_9\mathrm{F}/{}^{18}_{10}\mathrm{Ne}$ system and for ${}^{21}_9\mathrm{F}/{}^{21}_{10}\mathrm{Ne}/{}^{21}_{11}\mathrm{Na}/{}^{21}_{12}\mathrm{Mg}$.



Isospin conservation also imposes important constraints on strong interaction processes. Some examples follow:

a) Consider the reaction $d + d \rightarrow {}^4\text{He} + \pi^o$

The isospins of deuteron and ^{4}He are both zero, while the isospin of π^o is 1. The initial state $d + d$ can only have total isospin 0, while the final state has total isospin 1. Therefore, this reaction violates isospin conservation and can proceed only via electromagnetic interaction. An extensive effort to measure the cross-section of this reaction led to a determination of 0.8 pb/sr as an upper limit (using 800 MeV beam). Very recently, however, experimenters at Indiana University claimed a successful detection of this reaction.

b) The $\psi' \to {}^{J}\!/\!{}_{\psi} + \pi^0 + \pi^0$ was known to occur with a branching ratio of ~ 20% (meaning ~ 20% of all ψ' decays end up in this channel). In contrast, $\psi' \to {}^{J}\!/\!{}_{\psi} + \pi^0$ has a much smaller branching ratio of 0.1%. Since ${}^{J}\!/\!{}_{\psi}$ and ψ' are $c\bar{c}$ bound states with isospin = 0, the $\psi' \to {}^{J}\!/\!{}_{\psi} + \pi^0$ decay violates isospin conservation. This explains why this decay mode is much inhibited compared with the $\psi' \to {}^{J}\!/\!{}_{\psi} + \pi^0 + \pi^0$ mode, which does not violate isospin conservation.

c) Consider the reactions

a: $\pi^+ p \to \pi^+ p$	(elastic)
b: $\pi^- p \to \pi^- p$	(elastic)
c: $\pi^- p \to \pi^0 n$	(charge-exchange)

In these strong interactions, isospin is conserved. Now, consider the isospins of the following system:

$$\pi^+ p : \underbrace{|1,1\rangle}_{\pi^+} \underbrace{\left|{}^{1}\!/\!{}_{2}, {}^{1}\!/\!{}_{2}\right\rangle}_{p} = \underbrace{\left|{}^{3}\!/\!{}_{2}, {}^{3}\!/\!{}_{2}\right\rangle}_{\pi^+ p} \qquad \text{isospin}$$

where $|1,1\rangle$ signifies $|I, I_z\rangle$

$$\pi^- p : |1,-1\rangle \left|{}^{1}\!/\!{}_{2}, {}^{1}\!/\!{}_{2}\right\rangle = \sqrt{\frac{1}{3}}\left|{}^{3}\!/\!{}_{2}, -{}^{1}\!/\!{}_{2}\right\rangle - \sqrt{\frac{2}{3}}\left|{}^{1}\!/\!{}_{2}, -{}^{1}\!/\!{}_{2}\right\rangle$$

$$\pi^0 n : |1,0\rangle \left|{}^{1}\!/\!{}_{2}, -{}^{1}\!/\!{}_{2}\right\rangle = \sqrt{\frac{2}{3}}\left|{}^{3}\!/\!{}_{2}, -{}^{1}\!/\!{}_{2}\right\rangle + \sqrt{\frac{1}{3}}\left|{}^{1}\!/\!{}_{2}, -{}^{1}\!/\!{}_{2}\right\rangle$$

Now the cross-section is proportional to $\left|\sum_I M_I\right|^2$, namely $\sigma \sim \left|\sum_I M_I\right|^2$

where

$$M_{1/2} = \left\langle \psi_f \left(I = 1/2\right)\right| H \left|\psi_i \left(I = 1/2\right)\right\rangle$$
$$M_{3/2} = \left\langle \psi_f \left(I = 3/2\right)\right| H \left|\psi_i \left(I = 3/2\right)\right\rangle$$

we obtain

$$\sigma_a : \sigma_b : \sigma_c = \left|M_{3/2}\right|^2 : \frac{1}{9}\left|M_{3/2} + 2M_{1/2}\right|^2 : \frac{2}{9}\left|M_{3/2} - M_{1/2}\right|^2$$

If $M_{3/2} \gg M_{1/2}$, then $\sigma_a : \sigma_b : \sigma_c = 9:1:2$

If $M_{1/2} \gg M_{3/2}$, then $\sigma_a : \sigma_b : \sigma_c = 0:2:1$

d) As another example similar to c), we consider the following reactions:

$$k^- + p \to \pi^0 + \Lambda \text{ and } k^- + n \to \pi^- + \Lambda$$

Λ has isospin 0, and $\pi + \Lambda$ can only have isospin = 1. $k^- + p$ can couple to $I = 0$ and $I = 1$, since k^- has $I = 1/2$, $I_z = -1/2$ and p has $I = 1/2$, $I_z = 1/2$.

$$\left|1/2, -1/2\right\rangle \left|1/2, 1/2\right\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,0\rangle\right)$$

Similarly $k^- + n$ couples to $I = 1$ only

$$\left|1/2, -1/2\right\rangle \left|1/2, -1/2\right\rangle = |1,-1\rangle$$

One therefore has

$$\sigma\left(k^- + p \to \pi^0 + \Lambda\right) / \sigma\left(k^- + n \to \pi^- + \Lambda\right) = 1/2$$

e) In addition to the (p, n) isospin doublet, there are many other examples of isospin multiplets for mesons and baryons:

$I = 3/2$	Δ^{++}	Δ^{+}	Δ^{0}	Δ^{-}
(quark content)	(uuu)	(uud)	(udd)	(ddd)
I_z	$3/2$	$1/2$	$-1/2$	$-3/2$
mass	1230.8 MeV	1231.6	1233.5	

I = 1	Σ^{+}	Σ^{0}	Σ^{-}
(quark content)	(uus)	(uds)	(dds)
I_z	1	0	-1
Mass	1189.4 MeV	1192.6	1197.4

$I = 1/2$	Ξ^{0}	Ξ^{1}
(quark content)	(uss)	(dss)
I_z	$+1/2$	$-1/2$
mass	1314.8 MeV	1322.3 MeV

I = 1 (mesons)	π^{+}	π^{0}	π^{-}
(quark content)	$\left(u\bar{d}\right)$	$\left(u\bar{u}/d\bar{d}\right)$	$\left(\bar{u}d\right)$
I_z	+1	0	-1
mass	139.57 MeV	134.98 MeV	139.57 MeV

f) Finally, remember that isospin is not conserved in electromagnetic interaction. Although *p*, *n* have very similar mass, their magnetic moments are very different:

$$\mu_n = -1.91\mu_N \qquad \mu_p = 2.79\mu_N$$

This difference reflects the electromagnetic origins for the magnetic moment.

Conservation of Charge, Baryon Number, and Other Additive Quantum Numbers

Isospin symmetry is an example of SU(2) unitary symmetry. Extension to SU(3) can describe strong interaction (quantum chromodynamics) which is based on the symmetry of 3 colors (red, blue, green).
There are also important implications when the unitary symmetry is applied to the transformation in one dimension. In the U(1) symmetry, the unitary transformation is

$$\psi' = e^{i\alpha G}\psi$$

Note that there is no SU(1) symmetry, since the constraint of $\det\left(e^{i\alpha G}\right) = 1$ would completely fix the transformation to a trivial $\psi' = \psi$ transformation.

G is the generator of the U(1) transformation, and *G* is a conserved observable if the Hamiltonian *H* commutes with the U(1) transformation. Therefore, the energy eigenstate can also be an eigenstate for the generator *G*:

$$G\psi = q\psi \qquad H\psi = E\psi$$

The eigenvalue of *G* can be identified as the charge of a particle, for example. Charge conservation is therefore a consequence of U(1) symmetry.

Identical algebra can be adopted to describe other additive conservation laws such as baryons number conservation, lepton number conservation, strangeness conservation, etc. These conservation laws are derived based on the assumption that $[H, G] = 0$. Whether this is indeed true can only be tested by experiments. The charge conservation, however, is regarded as more robust compared with the other conservation laws (such as baryon number) since a dynamic theory, quantum electrodynamics, can be deduced based on the local gauge symmetry of U(1).

Just like the isospin symmetry in SU(2), the charge conservation in U(1) gives rise to additive quantum numbers. Consider a system consisting of *n* particles, the corresponding U(1) transformation is

$$e^{i\alpha G_1} e^{i\alpha G_2} \ldots.. e^{i\alpha G_n}$$
$$= e^{i\alpha(G_1 + G_2 + \ldots.. + G_n)}$$

Hence, $G = G_1 + G_2 + \ldots\ldots + G_n$ is a conserved observable.

We now consider baryon number conservation. This conservation law is motivated by the fact that all known reactions and decay processes involving baryons conserve baryon numbers. In particular, the lightest baryon, proton, appears to be perfectly stable.

We mentioned already that charge conservation is a result of a field (electromagnetic field) coupled to the electric charge. Lee and Yang suggested in 1955 that the apparent conservation of baryon numbers implies the existence of a long-range field coupled to baryon number (analogous to the case for charge conservation ↔ electromagnetic field). In particular, the gravitational force between an object and the earth

$$F_{\text{Gravity}} = K\frac{M_G M_{\text{earth}}}{r^2}$$

could contain an additional term sensitive to the total baryon number of the object

$$F_{\text{Baryon}} = K_B\frac{(M_N B)(M_N B_{\text{earth}})}{r^2}$$

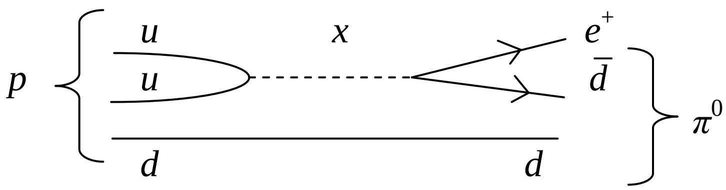
To measure such a new form of force coupled to baryon number (the fifth force), one could compare the gravitational force of two different objects having the same mass, but different total number of baryons. This is possible by selecting two objects made of different materials. The difference in the nuclear binding energies would give different total baryon number for equal mass. By a comparison of objects made of aluminum and platinum, it was found that

$$K_B < 10^{-9}\, K$$

Altough the standard model does not allow proton decay (none of the fermion-fermion-boson coupling diagrams we introduced in Chapter 1 allows proton decay), explanation for a well-known phenomenon in cosmology, namely the matter-antimatter asymmetry, requires baryon number non-conservation (the Sakharov conditions).

In Grand Unified Theories (GUT), such as SU(5), there exist “leptoquark” with mass $\simeq 10^{15}$ GeV. The leptoquark can couple to qq or ℓq.

A possible diagram for proton decay is



The most favorable decay channel in SU(5) is $p \rightarrow e^{+}\pi^{0}$. In supersymmetry GUT, the most favored decay channel is $p \rightarrow \bar{\nu}_{\tau} k^{+}$. Note that in these decays both baryon number conservation and lepton number conservation are violated. However, $B - L$, the difference between baryon number and lepton number, is conserved.

Conservation of strangeness is a broken symmetry only valid in strong and electromagnetic interactions. This conservation law is a special case of flavor conservation in strong and electromagnetic interactions. A related quantum number Y, called hypercharged, is defined as

$$Y = B + S \qquad (B \text{ is the baryon number and } S \text{ is the strangeness})$$

It is called hypercharge due to the following relationship between charge and Y:

$$Q = e\left(I_3 + \frac{1}{2}Y\right)$$

One can also define a generalized hypercharge Y' as

$$Y' = B + S + C + t + b$$

where c, b, t are the additive quantum number for charm, bottom, and top quarks.

We have

	B	**S**	**C**	**t**	**b**	**I_3**	**Y'**	$I_3 + \frac{1}{2}Y'$	**Q**
u	$\frac{1}{3}$	0	0	0	0	$+\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
d	$\frac{1}{3}$	0	0	0	0	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
c	$\frac{1}{3}$	0	+1	0	0	0	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
s	$\frac{1}{3}$	-1	0	0	0	0	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
t	$\frac{1}{3}$	0	0	+1	0	0	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
b	$\frac{1}{3}$	0	0	0	-1	0	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$

Note that the strangeness and bottom quantum numbers (*s*, *b*) are defined as -1 for the strange and bottom quarks, while *c* and *t* are +1 for the charm and top quarks. The above table shows that the relation

$$Q = e\left(I_3 + \frac{1}{2}Y'\right)$$

holds for all quarks (and for all antiquarks, where all additive quantum numbers change sign).

Note that charge conservation is valid for all types of interactions, while flavor (*s*, *c*, *t*, *b*) conservation only holds for strong and electromagnetic interactions. Furthermore, I_3 is conserved in strong and electromagnetic interactions (since Q and Y' are conserved). For weak interaction, Y' and I_3 are not conserved.

An example of I_3 non-conservation can be seen in the following weak decays

	$k^+ \rightarrow$	$\pi^+ +$	π^0	$\Lambda \rightarrow$	$p +$	π^-
I_3	$+\frac{1}{2}$	$+1$	0	0	$+\frac{1}{2}$	-1

Discrete Symmetry and Multiplicative Quantum Numbers

Space inversion (*P*), charge-conjugation (*C*), and time-reversal (*T*) are examples of discrete symmetry transformation which cannot be built up by infinitesimal transformation. As discussed earlier, there is no Hermitian generator for such discrete transformation. However, for *P* and *C* the transformation U_p and U_c are Hermitian and if the Hamiltonian commutes with U_p (U_c), the energy eigenstates could also be eigenstates of U_p (U_c). Since $U_p^2 = 1$, $U_c^2 = 1$, the possible eigenvalues are +1 and -1.

$$U_p \psi = \pm \psi = \lambda_p \psi$$

The multiplicative nature of the quantum number can be understood by considering a wave function consisting of several parts

$$\psi = \psi_1 \psi_2 \ldots\ldots \psi_n$$

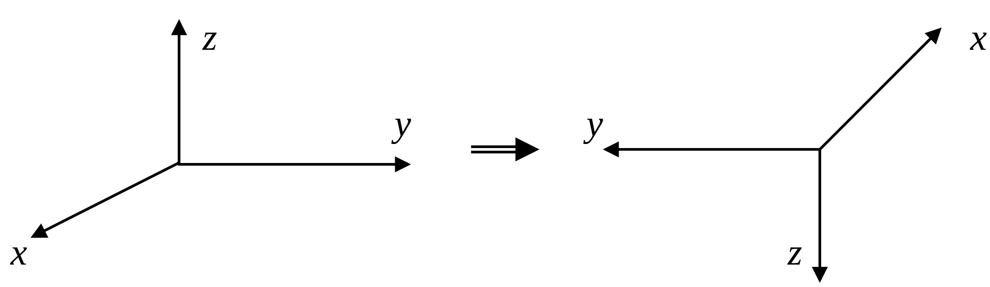
The corresponding parity transformation U_p is

$$\begin{aligned} U_p \psi &= U_p(1) U_p(2) \ldots\ldots U_p(n) [\psi_1 \psi_2 \ldots\ldots \psi_n] \\ &= \lambda_p(1) \lambda_p(2) \ldots\ldots \lambda_p(n) \psi \end{aligned}$$

Parity (*P*)

The parity operation corresponds to a transformation

$$(x, y, z) \rightarrow (-x, -y, -z)$$



It is clear that a right-handed coordinate system is changed into a left-handed system. Space-inversion is equivalent to mirror reflection followed by rotation, and space-inversion and mirror reflection are often treated interchangeably.

Consider a particle experiencing a central potential $v(r)$ which only depends on $r = |\vec{r}|$. The Schrödinger equation gives

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + v(r)\right]\psi(\vec{r}) = E\psi(\vec{r})$$

The eigenstate can be separated into the radial and angular parts

$$\psi(\vec{r}) = R_{n\ell}(r)Y_{\ell m}(\theta,\phi)$$

The $\vec{r} \to -\vec{r}$ can be expressed in spherical coordinates as

$$r \to r \qquad \theta \to \pi - \theta \qquad \phi \to \pi + \phi$$

Therefore,

$$\begin{aligned} U_p\psi(\vec{r}) = \psi(-\vec{r}) &= R_{n\ell}(r)Y_{\ell m}(\pi-\theta,\pi+\phi) \\ &= R_{n\ell}(r)(-1)^\ell Y_{\ell m}(\theta,\phi) \\ &= (-1)^\ell \psi(\vec{r}) \end{aligned}$$

For even ℓ, parity = +1
For odd ℓ, parity = -1

The transformations of various quantities under space inversion are as follows:

$\vec{p} \to -\vec{p}$ (momentum) $\qquad \vec{L}(=\vec{r}\times\vec{p}) \to L$ (angular momentum)

$\vec{s} \to \vec{s}$ (spin) $\qquad \vec{E} \to -\vec{E}$ (electric field)

$\vec{B} \to \vec{B}$ (magnetic field)

The transformation properties of $\vec{E}$ and $\vec{B}$ field under parity can be understood by requiring that the equation of motion for a charged particle

$$\vec{F} = m\frac{d^2}{dt^2}\vec{r} = e\left[\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right]$$

be invariant under space inversions (both $\vec{r}$ and $\vec{v}$ charge sign, hence $\vec{E} \to -\vec{E},\ \vec{B} \to \vec{B}$).

What are the intrinsic parity of various hadrons (mesons and baryons), leptons, and gauge bosons?

First consider mesons. Since a meson consists of a quark and an antiquark, we have

$$\text{parity (meson)} = \text{parity } (q) \times \text{parity } (\bar{q}) \times (-1)^L$$

where L is the orbital angular momentum between the quark (q) and antiquark $(\bar{q})$.

The intrinsic parity of quarks is defined as +1, and the antiquark's parity is -1, opposite to that of the quark.

Hence $$\text{parity (meson)} = (-1)^{L+1}$$

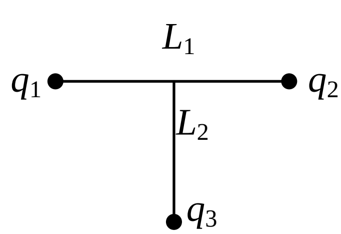
As will be discussed later, the lightest mesons are mostly $L = 0$ states (i.e. π, k, η, ρ, ω, ϕ . . .). Therefore, these mesons have negative parity ($\lambda = -1$).

Mesons with $L = 1$ (a_0, f_0, b, h, etc.) have positive parity ($\lambda = +1$).

For baryons consisting of three quarks, we have

$$\text{parity (baryon)} = \text{parity } (q_1) \times \text{parity } (q_2) \times \text{parity } (q_3) \times (-1)^{L_1} \times (-1)^{L_2}$$

where L_1 is the orbital angular momentum between q_1 and q_2, and L_2 is the orbital angular momentum between q_3 and the center-of-mass of q_1q_2:



For light baryons, such as p, n, Λ, Σ, Ξ, Ω, $L_1 = 0$, $L_2 = 0$ and $\lambda = +1$. Note that if the intrinsic parity of quarks were defined as negative, then these baryons would have $\lambda = -1$. For baryons with $L_1 + L_2$ = odd, their parity is negative.

The intrinsic parity of photon is negative. Photon has spin = 1, and it exists in two states:

λ = +1 for positive helicity state (right-handed), or
λ = -1 for negative helicity state (left-handed)

The wave function can be written as

$$A_{\lambda=+1} = -\frac{1}{\sqrt{2}}\left(\vec{e}_x + i\vec{e}_y\right)e^{ipz}$$
$$A_{\lambda=-1} = \frac{1}{\sqrt{2}}\left(\vec{e}_x - i\vec{e}_y\right)e^{ipz}$$

for photons moving along the *z*-axis. Upon parity transformation, $A_{\lambda=+1}$ becomes $-A_{\lambda=-1}$ and vice versa.

We now consider neutrinos. It turns out that neutrinos are not eigenstates of parity. Upon parity transformation, a left-handed neutrino would become a right-handed neutrino. Since right-handed neutrino is not found in nature (only right-handed antineutrino exists), neutrino is not a parity eigenstate.

<u>Tests of Parity Conservation</u>

a) *Atomic Physics*

In 1924, Laporte found the following peculiar phenomenon in atomic x-ray transitions in iron atoms. The atomic levels can be separated into two groups, and photon transitions were only observed between two states belonging to the two different groups. No transitions between states within the same group were observed. In 1927, soon after the discovery of quantum mechanics, Wigner suggested that the ‘Laporte’s Rule’ is a consequence of parity conservation in atomic transitions.

The probability for an atom to make a transition from a state ψ_a to a state ψ_b with the emission of electric dipole radiation is proportional to the square of the matrix element

$$\langle \psi_b | \vec{d} | \psi_a \rangle$$

where $\vec{d} = \sum_i e\vec{r}_i$ is the electric dipole moment operator summed over all electrons. This matrix element is invariant under parity transformation if parity is conserved:

$$\langle \psi_b | \vec{d} | \psi_a \rangle = \langle \eta_b \psi_b | -\vec{d} | \eta_a \psi_a \rangle = -\eta_a \eta_b \langle \psi_b | \vec{d} | \psi_a \rangle$$

where η_a, η_b are the parities of ψ_a and ψ_b.

If $\eta_a \eta_b = +1$, then $\langle \psi_b | \vec{d} | \psi_a \rangle = -\langle \psi_b | \vec{d} | \psi_a \rangle = 0$

Hence, dipole transitions can only occur between states which have opposite parities, if parity is conserved.

An interesting extension of the Laporte's Rule concerns the static dipole moment of an atom, a nucleus, or other elementary particle. In this case, $\psi_a = \psi_b$ and $\eta_a \eta_b = +1$. Hence, electric dipole moment vanishes if parity is conserved.

Since atomic physics is dominated by electromagnetic interaction, and since parity is conserved in electromagnetic interaction, it is not surprising that no significant parity violation has been observed in atomic physics. As will be discussed later, the interference of electromagnetic and weak interaction does allow parity violation in atomic physics. The effect is tiny, and could only be detected in very sensitive measurements.

b) *Parity Violation Search in Strong Interaction*

Some examples of parity violation search in nuclear physics, where strong interaction dominates, include

$$^{20}\text{Ne}(1^+, \text{Ex} = 11.3\text{MeV}) \rightarrow {}^4\text{He}(0^+) + {}^{16}\text{O}(0^+)$$

This α-decay of an excited ^{20}Ne 1^+ state to ^{16}O ground state violates parity. Conservation of angular momentum requires that L = 1 between ^{4}He and ^{16}O. The total parity for ^{4}He + ^{16}O is therefore negative (due to the $(-1)^L$ factor).

$$d(1^+) + {}^4\text{He}(0^+) \rightarrow {}^6\text{Li}\ (0^+, \text{Ex} = 3.6\ \text{MeV})$$

This reaction also violates parity, since angular momentum conservation dictates that the orbital angular momentum L between d and ^{4}He be 1.

Neither reaction has been observed experimentally, showing that parity is conserved in strong interaction.

c) *Parity Violation in Weak Interaction*

- The τ-θ puzzle

Prior to 1956, the θ- and τ-mesons (not to confuse with the τ-lepton) were found to have the decay modes

$$\theta^+ \rightarrow \pi^+ + \pi^0$$
$$\tau^+ \rightarrow \pi^+ + \pi^0 + \pi^0 \quad (\text{and } \tau^+ \rightarrow \pi^+ + \pi^- + \pi^+)$$

The masses and lifetimes of θ and τ were equal within experimental errors. Assuming θ, τ have spin = 0, the $\theta^+ \rightarrow \pi^+ + \pi^0$ decay requires that $\pi^+ + \pi^0$ has positive-parity (since $\pi^+ + \pi^0$ parity = $(-1)\bullet(-1)\bullet(-1)^L = +1$, $L = J = 0$). Similarly, one can show that τ^+ has to have negative parity. It was a great puzzle why two particles (θ, τ) have identical masses and lifetimes, but opposite intrinsic parities.

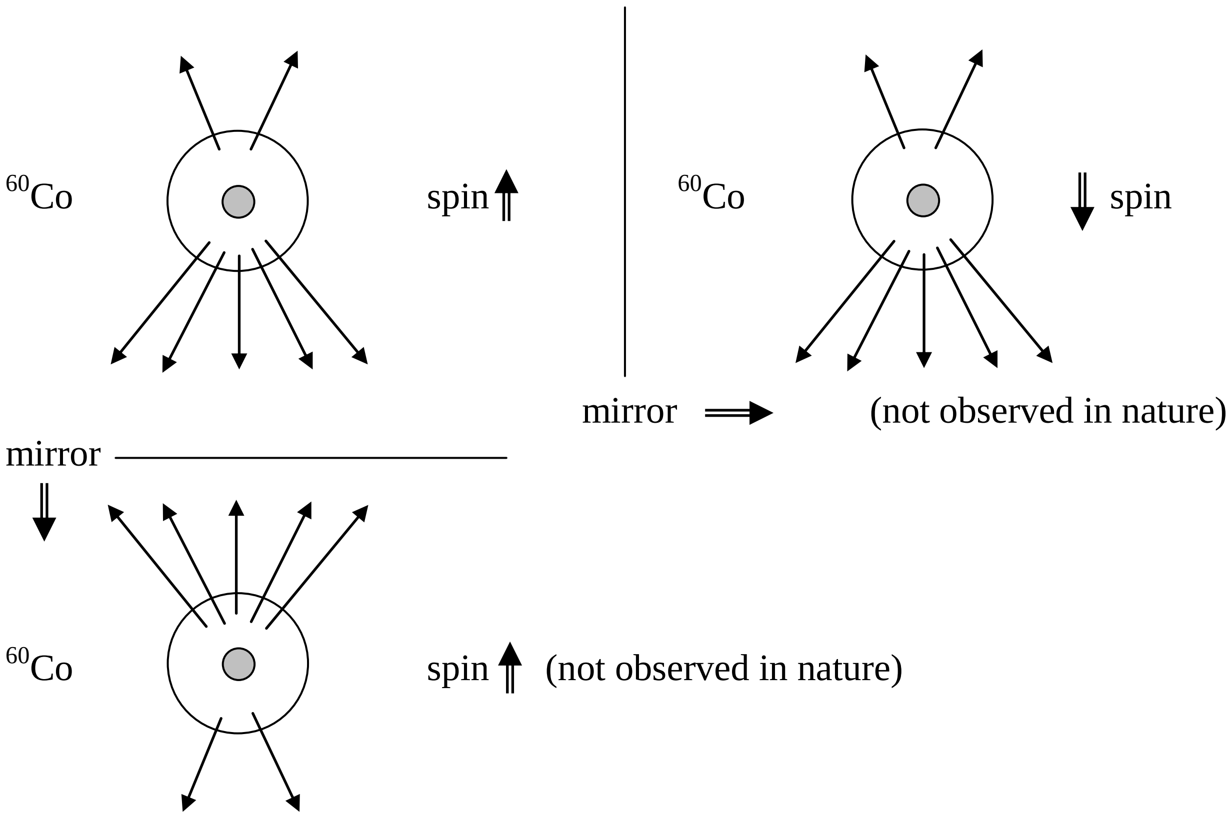
- The ^{60}Co Experiment

Lee and Yang suggested that the τ-θ puzzle could be resolved by parity non-conservation in weak interaction. Although there was plenty of evidence for the validity of parity conservation in electromagnetic (atomic) and strong (nuclear) interactions, Lee and Yang pointed out that there was no experimental evidence whatsoever for parity conservation in nuclear β-decays or other weak decays of mesons and hyperons.

Wu, Ambler, Hayward, Hoppes and Hudson performed an experiment which conclusively demonstrated parity violation in nuclear β-decay. They used polarized ^{60}Co nuclei and measured electrons from the following decay:

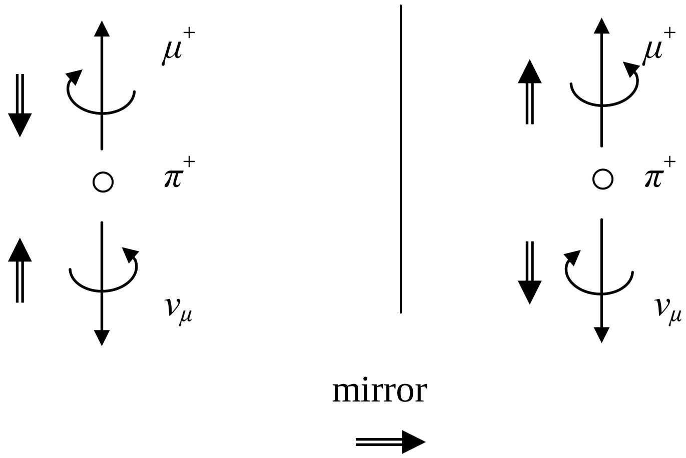
$$^{60}\text{Co} \rightarrow ^{60}\text{Ni} + e^{-} + \bar{\nu}_e$$

They found e^- prefers to be emitted in the direction opposite to the ^{60}Co spin direction:



From the above figure, it is clear that the mirror images give un-physical situations, where e^- prefers to be emitted in the direction along the ^{60}Co spin direction (independent of how you place the mirror).

Upon hearing the news on Wu's experiment, Garwin, Lederman, and Weinrich carried out an elegant experiment confirming large parity violation effect in π^+ and μ^+ decays. They found 1) μ^+ in the $\pi^+ \rightarrow \mu^+ + \nu_\mu$ decay is polarized, i.e. the expectation value $\left\langle \vec{s} \bullet \vec{p}_{\mu^+} \right\rangle \neq 0$; and 2) e^+ emitted in the $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$ is not isotropic and the preferred direction is correlated with the spin orientation of μ^+.



(not observed in nature)

Note that the asymmetry in the angle of emission of e in $^{60}\overrightarrow{\mathrm{Co}} \to {}^{60}\mathrm{Ni} + e^- + \bar{\nu}_e$ decay and the $\vec{\mu}^+ \to e^+ + \nu_e + \bar{\nu}_\mu$ both reflect the existence of a non-vanishing expectation value for

$$\langle \vec{s} \bullet \vec{p}_e \rangle$$

where $\vec{s}$ is the spin of ^{60}Co (or μ^+) and $\vec{p}_e$ is the momentum of e. Since $\vec{s} \bullet \vec{p}_e$ changes sign under parity transformation, it cannot have non-zero expectation value if parity is conserved. Similarly, in the $\pi^+ \to \mu^+ + \nu_\mu$ decay, the fact that μ^+ is longitudinally polarized means that the expectation value for

$$\left\langle \vec{s}_{\mu^+} \bullet \vec{p}_{\mu^+} \right\rangle$$

is not zero. Again $\vec{s}_{\mu^+} \bullet \vec{p}_{\mu^+}$ changes sign under parity and must vanish if parity is conserved.

Quantities such as $\vec{s} \bullet \vec{p}$ are called pseudoscalars. They behave like a scalar under rotation, but changes sign under parity transformation. Quantities such as $\vec{L}$ and $\vec{s}$ are axial vectors, which behave like a vector under rotation, but do not change sign under parity.

We can revisit the τ-θ puzzle and observe that the

$$k^+ \to \pi^+ + \pi^0$$

decay manifestly violates parity. Since k^+, π^+, π^0 all have J^π (J is spin, π is parity) = 0^-, the parity for $\pi^+ + \pi^0$ is (-1) x (-1) x $(-1)^L$. Conservation of angular momentum requires $L = 0$. Hence, the parity for $\pi^+ + \pi^0$ is +1, opposite to the parity of the initial particle k^+.

Charge-Conjugation

The charge-conjugation transformation, U_c, reverses the sign of additive quantum numbers. Other quantum number related to space-time (energy, momentum, spin) remain the same under U_c.

$$U_c\psi(Q,\ B,\ S,\ L) = \psi(-Q,\ -B,\ -S,\ -L)$$

(Q, B, S, L are charge, baryon number, strangeness, lepton number)

Since $U_c^2 = 1$, the eigenvalues for U_c are +1, -1. From the above relation, it is clear that only particles with $Q = 0$, $B = 0$, $S = 0$, $L = 0$ could be eigenstates of the c-parity. Particles such as π^0, ρ^0, γ have definite intrinsic c-parity.

Although very few particles are themselves eigenstates of c, it is possible to form c-parity eigenstates by considering systems consisting of several particles. For example, ($\pi^+\pi^-$), $(p,\ \bar{p})$, (π^+, π^-, π^0) are eigenstates of c-parity.

What is the c-parity of a system consisting of a boson and an antiboson (such as $\pi^+\pi^-$)?

$$\psi = \psi\text{ (space) }\psi\text{ (spin) }\psi\text{ (intrinsic)}$$

A particle-antiparticle exchange is identical to interchanging the two particles. Let L, S be the orbital angular momentum and the total spin of the two bosons. Upon particle-antiparticle interchange, one obtains a factor $(-1)^L$ from ψ (space), $(-1)^S$ from ψ (spin) and the c-parity of boson-antiboson pair is

$$(-1)^{L+S}$$

For a fermion-antifermion pair (like $p\bar{p}$ or e^+e^-), one obtains $(-1)^L$ from ψ (space), $(-1)^{S+1}$ from ψ (spin) and (-1) from ψ (intrinsic). The overall c-parity is again $(-1)^{L+S}$.

The charge parity, η_c, for neutral mesons which have $L = 0$, $S = 0$, is therefore +1 (π^0, η^0). For ρ^0, ϕ mesons which have $L = 0$, $S = 1$, η_c is -1.

The photon has $\eta_c = -1$ since all components of the electromagnetic field change sign under charge conjugation. A system of n photons has $\eta_c = (-1)^n$.

e^+e^- can form bound states called positronium. The shortest-lived state of positronium is in a ^1S_o state ($S = 0$, $L = 0$, $J = 0$) which can decay into 2γ (since $\eta_c = (-1)^{L+S} = +1$)

$$e^+e^- \, (^1S_o) \rightarrow 2\gamma$$

Note that c-parity conservation dictates that $e^+e^- \rightarrow 2\gamma$ has to occur from a $\eta_c = +1$ positronium state. Similarly,

$$e^+e^- \, (^3S_1) \rightarrow 3\gamma$$

Note that the positronium ^1S_o state is analogous to π^0, which consists of a $q\bar{q}$ pair in ^1S_o state. Indeed, the dominant decay mode for π^0 is

$$\pi^0 \rightarrow 2\gamma$$

while the $\pi^0 \rightarrow 3\gamma$ decay mode, which violates c-parity, has never been observed.

It can be readily verified that c-parity conservation (in strong and electromagnetic interactions) forbid the following decays:

$$\begin{gathered} \eta \rightarrow 3\gamma \\ \eta \rightarrow \pi^0\gamma \\ \rho \rightarrow \pi^0\pi^0\pi^0 \\ \rho \rightarrow \eta\pi^0 \end{gathered}$$

<u>Tests of *c*-invariance</u>

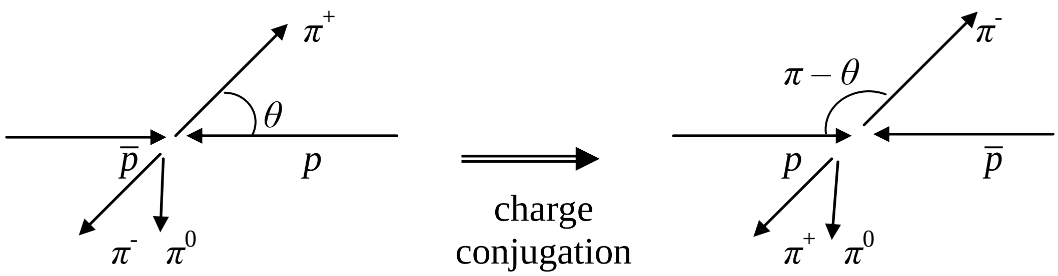
A direct test of c-invariance can be made by comparing the cross-sections, energy distribution for a reaction

$$a + b \rightarrow c + d + e + \ldots$$

with the charge conjugation reaction

$$\bar{a} + \bar{b} \rightarrow \bar{c} + \bar{d} + \bar{e} + \ldots.$$

An example is the $\bar{p}p \rightarrow \pi^+\pi^-\pi^0$ reaction. In the center-of-mass frame:



Invariance under charge conjugation requires

$$\frac{d\sigma}{d\Omega}(\theta)_{\pi^+} = \frac{d\sigma}{d\Omega}(\pi - \theta)_{\pi^-}$$

i.e. the cross-section for π^+ to be produced at an angle θ (with respect to the $\bar{p}$ direction) is identical to the cross-section for π^- to be produced at the angle $\pi - \theta$. In a similar fashion, one can readily show that

$$\frac{d\sigma}{d\Omega}(\theta)_{\pi^0} = \frac{d\sigma}{d\Omega}(\pi - \theta)_{\pi^0}$$

In other words, the π^0 angular distribution in the center-of-mass frame should be symmetric about $\theta = 90^o$.

No evidence for c-parity violation was observed in $\bar{p}p \rightarrow \pi^+\pi^-\pi^0$ reaction. Extensive study of a similar process

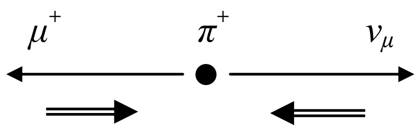
$$\eta \rightarrow \pi^+\pi^-\pi^0$$

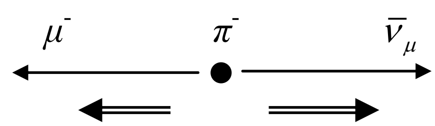
also did not find any evidence for *c*-violation.

c-violation in Weak Interaction

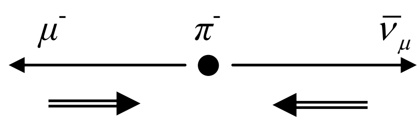
Parity violation in weak interaction also leads to c-violation. For example, neutrino is left-handed and antineutrino is right-handed. This fact alone shows that charge-conjugation symmetry is violated, since the charge-conjugation operation on left-handed neutrino would lead to a left-handed antineutrino.

Another example is provided by the charged pion decay. $\pi^+ \to \mu^+ + \nu_\mu$ decay produces a left-handed ν_μ and μ^+, while $\pi^- \to \mu^- + \bar{\nu}_\mu$ produces right-handed $\bar{\nu}_\mu$ and μ^- (the double-arrow indicates the spin direction):



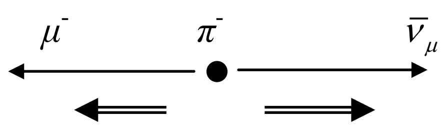


A charge-conjugation operation on $\pi^+ \to \mu^+ + \nu_\mu$ would lead to



which is not observed in nature.

Therefore, the $\pi^+ \to \mu^+ + \nu_\mu$ decay violates c-parity. It is interesting to note that the CP operation, which combines the c and p operations, is invariant. The combined CP operation on $\pi^+ \to \mu^+ + \nu_\mu$ would yield



which is observed in nature.

CP-violation

After the discovery of parity violation in 1957, it was immediately recognized that c-parity is also violated in weak interaction. However, it was generally believed that the combined operation, CP, was a valid symmetry. In 1964, CP-violation was discovered in neutral *k*-meson decays. The physics origin of CP violation is still poorly understood, and it remains a very active area of research in particle and nuclear physics.