Chapter 4

Dirac Equation

The non-relativistic Schrödinger equation was obtained by noting that the Hamiltonian

$$H = {P^2}/{2m} \tag{4.1}$$

can be transformed into an operator form with the substitutions

$$H \to i\frac{\partial}{\partial t} \quad (\hbar = 1,\ c = 1) \tag{4.2}$$
$$P \to -i\vec{\nabla}$$

resulting

$$i\frac{\partial}{\partial t}4(x,\ t) = \frac{\nabla^2}{2m}4(x,\ t) \tag{4.3}$$

This equation is linear in the time-derivative and quadratic in the space derivative, and is manifestly non-covariant under Lorentz transformation. Clearly, the left-hand side and the right-hand side of Equation 4.3 transforms differently under Lorentz transformation.

We briefly review Lorentz transformation.

Space-time coordinates (t, x, y, z) are denoted by the contravariant 4-vector

$$x^\mu \equiv (x^o,\ x^1,\ x^2,\ x^3) \equiv (t,\ x,\ y,\ z\ )$$

The covariant 4-vector, x_μ, is

$$x_\mu \equiv (x_o,\ x_1,\ x_2,\ x_3) \equiv (t,\ \text{-}x,\ \text{-}y,\ \text{-}z) = g_{\mu\nu}x^\nu$$

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.4}$$

$g_{\mu\upsilon}$ is the metric tensor

The Lorentz transformation relates $x^{\mu\prime}$ to x_μ:

$$\left(x^{\mu}\right)' = \Lambda^{\mu}_{\nu} x^{\nu} \tag{4.5}$$

For an inertial system S' moving along z with $v/c = \beta$, we have

$$\Lambda = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \tag{4.6}$$

where $\gamma = \left(1-\beta^2\right)^{-1/2}$

Any set of four quantities which transform like x^μ under Lorentz transformation is called a four-vector. The total energy E and the momentum $\vec{P}$ form a four-vector

$$P^{\mu} = \left(P^{o},\ P^{1},\ P^{2},\ P^{3}\right) = \left(\mathrm{E},\ \vec{P}\right) \tag{4.7}$$

Scalar product of two 4-vectors are invariant under Lorentz transformation

$$A_{\mu} B^{\mu} = g_{\nu\mu} A^{\mu} B^{\nu} \tag{4.8}$$

Some examples:

$$x \bullet x = x_{\mu} x^{\mu} = t^2 - x^2$$
$$P \bullet P = P_{\mu} P^{\mu} = \mathrm{E}^2 - P^2$$
$$x \bullet P = x_{\mu} P^{\mu} = t\ \mathrm{E} - \vec{x} \bullet \vec{P}$$

The $H \to i\frac{\partial}{\partial t},\ \vec{P} \to -i\vec{\nabla}$ transcription is Lorentz covariant

$$P^{\mu} \to i\frac{\partial}{\partial x_{\mu}} = i\partial^{\mu}$$

One can readily obtain the continuity equation from the Schrödinger equation:

$$\frac{\partial \zeta}{\partial t} + \vec{\nabla} \bullet \vec{J} = 0 \tag{4.9}$$

where $\zeta = \psi^* \psi \geq 0,\ \vec{J} = -\frac{i}{2m}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$

ζ is the probability density, and $\vec{J}$ is the current density.

Note that the continuity equation can be written in a covariant form

$$\partial_\mu j^\mu = 0,\ \ j^\mu = \left(\zeta,\ \vec{J}\right) \tag{4.10}$$

The early attempt for formulating a Lorentz covariant equation started from the relativistic energy-momentum relation

$$\mathrm{E}^2 = P^2 + m^2$$

with the $\mathrm{E} \rightarrow i\frac{\partial}{\partial t},\ \ \vec{P} \rightarrow -i\vec{\nabla}$ transcription, one obtains

$$-\frac{\partial^2}{\partial t^2}\psi = \left(-\nabla^2 + m^2\right)\psi \tag{4.11}$$

which is the relativistic Schrödinger equation, or the Klein-Gordon equation.

Equation 4.11 can be written as

$$\left(\square^2 + m^2\right)\psi = 0 \tag{4.12}$$

where
$$\square^2 = \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \partial_\mu \partial^\mu \tag{4.13}$$

is the d'Alembertian operator, a Lorentz invariant operator.

From the Klein-Gordon equation, one can readily derive the following continuity equation:

$$\frac{\partial \zeta}{\partial t} + \vec{\nabla} \bullet \vec{J} = 0$$

$$\left.\begin{aligned} \zeta &= i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right) \qquad (4.14) \\ \vec{J} &= -i\left(\phi^* \vec{\nabla} \phi - \phi \vec{\nabla} \phi^*\right) \qquad (4.15) \end{aligned}\right\} j^\mu = i\left(\phi^* \partial^\mu \phi - \phi\, \partial^\mu \phi^*\right)$$

For a plane-wave solution to Klein-Gordon equation

$$\phi = N e^{-i\left(\mathrm{E}t - \vec{P} \bullet \vec{x}\right)} \tag{4.16}$$

We have

$$\zeta = 2E|N|^2, \quad \vec{J} = 2\vec{P}|N|^2 \tag{4.17}$$

Since $\mathrm{E} = I\left(P^2 + m^2\right)^{1/2}$, ζ can be negative for the E < 0 solutions.

The origin of this problem is the second derivative in time term $\partial^2 / \partial t^2$.

Dirac tried to find a relativistic covariant equation with a positive definite probability density. He assumed an equation linear in $\frac{\partial}{\partial t}$ and in $\vec{\nabla}$:

$$\begin{aligned} H\psi &= \left(\vec{\alpha} \bullet \vec{P} + \beta m\right)\psi \\ \vec{\alpha} \bullet \vec{P} &= \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 \end{aligned} \tag{4.18}$$

It is evident that α's and β cannot be just numbers, because the equation would not be invariant even under spatial rotation. Dirac proposed that the equation be considered as a matrix equation. It should satisfy $H^2\psi = \left(P^2 + m^2\right)\psi$.

Applying H to Equation 4.18 leads to

$$H^2\psi = \left[\sum_i \alpha_i^2 P_i^2 + \sum_{i \neq j} \left(\alpha_i \alpha_j + \alpha_j \alpha_i\right) P_i P_j + \sum_i \left(\alpha_i \beta + \beta \alpha_i\right) m P_i + \beta^2 m^2\right]\psi$$

Therefore

$$\{\alpha_i,\ \alpha_j\} = 2\delta_{ij}\ ;\ \{\alpha_i,\ \beta\} = 0$$
$$\alpha_i^2 = \beta^2 = 1$$

What are the other properties of α_i, β?
First, α_i, β are Hermitian, since H is Hermitian (see Equation 4.18).

In addition, $Tr(\alpha_i) = 0$, $Tr(\beta) = 0$

This can be shown by the anticommutator relations in Equation 4.19. Recall $Tr(AB) = Tr(BA)$

$$Tr(\alpha_i) = Tr(\alpha_i\beta^2) = Tr(\beta\alpha_i\beta) = Tr(-\alpha_i\beta\beta)$$
$$= -Tr(\alpha_i) \Rightarrow Tr(\alpha_i) = 0 \qquad (4.20)$$

Similarly, $Tr(\beta) = 0$
Therefore, α_i, β are traceless, Hermitian matrices.

The eigenvalues of α_i, β can only be +1 or -1 since $\alpha_i^2 = \beta^2 = 1$.

$$\alpha x = \lambda x \Rightarrow \alpha^2 x = \lambda^2 x = x \Rightarrow \lambda = \pm 1 \qquad (4.21)$$

The trace of a matrix is equal to the sum of all its eigenvalues. For a traceless matrix having +1, -1 as possible eigenvalues, it must have an even number of dimension n. n cannot be 2, since only three traceless $n = 2$ independent matrices exist $(\sigma_x,\ \sigma_y,\ \sigma_z,\ \text{for example})$.

$n = 4$ is the smallest possible dimension for α_i, β. The Pauli-Dirac representation is given as

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (4.22)$$

The Dirac equation $\left[\vec{\alpha}\bullet\vec{P} + m\beta\right]\psi = \mathrm{E}\psi$ can be expressed in a more covariant fashion by multiplying β from the left:

$$\left(\beta\mathrm{E} - (\beta\vec{\alpha})\bullet\vec{P} - m\right)\psi = 0 \qquad (4.23)$$

Introducing $\gamma^\mu = (\beta,\ \beta\vec{\alpha})$, Equation 4.23 becomes

$$\left(\gamma^\mu P_\mu - m\right)\psi = 0 \tag{4.24}$$

$P_\mu = i\partial_\mu$, therefore

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0 \qquad \left(\text{recall } \partial_\mu = \frac{\partial}{\partial x^\mu}\right) \tag{4.25}$$

Explicitly,

$$\gamma^o = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{4.26}$$

γ^μ's satisfy the anticommutation relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I \tag{4.27}$$

$$\left(\gamma^o\right)^2 = I, \qquad \left(\gamma^i\right)^2 = -I$$

Although γ^o is Hermitian, $(\gamma^o)^+ = \gamma^o$, γ^i is anti-Hermitian:

$$(\gamma^i)^+ = -\gamma^i \tag{4.28}$$

However,

$$(\gamma^\mu)^+ = \gamma^o \gamma^\mu \gamma^o \quad \text{and} \quad \gamma^\mu = \gamma^o (\gamma^\mu)^+ \gamma^o \tag{4.29}$$

To obtain the adjoint Dirac equation, one takes Hermitian conjugate of Equation 4.25

$$-i\partial_\mu \psi^+ \left(\gamma^\mu\right)^+ - m\psi^+ = 0 \tag{4.30}$$

Since γ^μ is not Hermitian for $\mu = 1, 2, 3$, it is useful to multiply Equation 4.30 from the right by γ^o to yield (using Equation 4.29)

$$-i\partial_\mu \left(\psi^+ \gamma^o\right)\gamma^\mu - m\left(\psi^+ \gamma^o\right) = 0 \tag{4.31}$$

Define $$\bar{\psi} = \psi^+ \gamma^o \tag{4.32}$$

the adjoint Dirac equation is $$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0 \tag{4.33}$$

or $$\bar{\psi}\left(\not{P} + m\right) = 0 \tag{4.34}$$

where $$\not{P} = \gamma^{\mu} P_{\mu}$$

and P acts to its left in Equation 4.34.

One is now ready to check the continuity equation and the probability density and current density for the Dirac equation.

Multiplying Equation 4.25 by $\bar{\psi}$ to its left, and Equation 4.33 by ψ to its right, we obtain

$$\begin{aligned} i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \bar{\psi}m\psi = 0 \\ i\partial_{\mu}\bar{\psi}\gamma^{\mu}\psi + m\bar{\psi}\psi = 0 \end{aligned} \tag{4.35}$$

Adding these two equations yields

$$\partial_{\mu}\left(\bar{\psi}\gamma^{\mu}\psi\right) = 0 \tag{4.36}$$

Define $$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi \tag{4.37}$$

the continuity equation $\partial_{\mu} j^{\mu} = 0$ follows.

Note that $$\zeta = j^{o} = \bar{\psi}\gamma^{o}\psi = \psi^{+}\psi = \sum_{i=1}^{4}\left|\psi_{i}\right|^{2} \geq 0$$

Hence the probability density is indeed positive definite for the Dirac equation.

As will be shown later j^{μ} transforms like a 4-vector, and $\zeta = j^{o}$ is not a Lorentz invariant. Rather, it is the time component of a 4-vector. It is not too surprising that ζ is not a Lorentz invariant, since a probability density satisfies the following:

$$\int \zeta \alpha^{3} x \ = \ \text{const} \tag{4.38}$$

Since $\alpha^{3}\chi$ undergoes Lorentz contraction, it is natural that ζ must also change under the Lorentz transformation in order to preserve the integral.

Each component of the four Dirac wave function components satisfies the Klein-Gordon equation. Multiplying Equation 4.25 from the left by $i\gamma^{\mu}\partial_{\mu}$, we have

$$-\gamma^{\mu}\gamma^{\nu}\partial_{\mu}\partial_{\nu}\psi - m^2\psi = 0 \tag{4.39}$$

or

$$\left[\frac{1}{2}\left(\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu}\right)\partial_{\mu}\partial_{\nu} + m^2\right]\psi = 0$$

hence

$$\left(g^{\mu\nu}\partial_{\mu}\partial_{\nu} + m^2\right)\psi = 0$$

or

$$\left(\partial_{\mu}\partial^{\mu} + m^2\right)\psi = 0 \tag{4.40}$$

Note that Equation 4.40 is understood as four separate uncoupled equations for each component of ψ.

We turn next to the solutions to the Dirac equation for a free particle.

Because of Equation 4.40, the Dirac equation admits a free-particle solution as follows

$$\psi = U(P)e^{-iP\bullet x} \tag{4.41}$$

where $U(P)$ is independent of x and is a 4-component spinor. Substituting Equation 4.41 into Equation 4.25 gives

$$\left(\gamma^{\mu}P_{\mu} - m\right)U = 0, \quad \text{or} \quad \left(\not{P} - m\right)U = 0 \tag{4.42}$$

Similarly, the adjoint equation is readily obtained:

$$\bar{U}\left(\not{P} - m\right) = 0 \tag{4.43}$$

For a Dirac particle at rest, $\vec{p} = 0$, and Equation 4.42 becomes

$$\left(\gamma^{o}\mathrm{E} - m\right)U = 0 \qquad \left(\psi = Ue^{-i\mathrm{E}t}\right) \tag{4.44}$$

or

$$EU = m\gamma^o U = \begin{pmatrix} m & & & \\ & m & & \\ & & -m & \\ & & & -m \end{pmatrix} U \tag{4.45}$$

The eigenvectors are

$$U_1 = N\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad U_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad U_3 = N\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad U_4 = N\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.46}$$

with eigenvalues *m*, *m*, -*m*, -*m*, respectively.
N is the hormalization factor.

To obtain the solutions for $\vec{p} \neq 0$, one can either solve the Dirac Equation directly, or by boosting the $\psi(p=0)$ solution to another inertia frame moving with $-\vec{p}$ with respect to the rest frame.

It is instructive to examine the method involving the Lorentz boost. To use this method, we need to first address the question – How does ψ transform under Lorentz transformation?

Let us suppose that in reference frame *S*, the Dirac equation is written as

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0 \tag{4.47}$$

In another frame S′, where $x^{\mu\prime} = \Lambda^\mu_\nu x^\nu$, the Dirac equation becomes

$$\left(i\gamma^\mu \frac{\partial}{\partial x^{\mu\prime}} - m\right)\psi'(x') = 0 \tag{4.48}$$

Now,

$$\psi'(x') = \psi'(\Lambda x) = S\psi(x) \tag{4.49}$$

and

$$\psi(x) = S^{-1}\psi'(x') \tag{4.50}$$

Substituting Equation 4.50 into Equation 4.47 and noting

$$\frac{\partial}{\partial x^{\mu}} = \frac{\partial}{\partial x^{\nu\prime}}\frac{\partial x^{\nu\prime}}{\partial x^{\mu}} = \Lambda^{\nu}_{\mu}\frac{\partial}{\partial x^{\nu\prime}} \tag{4.51}$$

Equation 4.47 becomes, upon multiplying *S* from the left,

$$\left(iS\gamma^{\mu}\Lambda^{\nu}_{\mu}\frac{\partial}{\partial x^{\nu\prime}}S^{-1} - m\right)\psi'(x') = 0 \tag{4.52}$$

Requiring Equation 4.52 to be identical to Equation 4.48 gives

$$iS\gamma^{\mu}\Lambda^{\nu}_{\mu}\frac{\partial}{\partial x^{\nu\prime}}S^{-1} = i\gamma^{\mu}\frac{\partial}{\partial x^{\mu\prime}}$$

or

$$S\gamma^{\mu}\Lambda^{\nu}_{\mu}S^{-1} = \gamma^{\nu}$$

Equivalently,

$$\Lambda^{\nu}_{\mu}\gamma^{\mu} = S^{-1}\gamma^{\nu}S$$

Our task is to find *S*, which is a 4 x 4 matrix, for any Lorentz transformation Λ, such that Equation 4.53 is satisfied.

We consider three cases of Lorentz transformation:

1) Lorentz boost
2) Spatial rotation
3) Space inversion (parity operation)

The cases 1) and 2) can be built up with infinitesimal Lorentz transformations.

Consider an infinitesimal Lorentz transformation

$$\Lambda^{\nu}_{\mu} = \delta^{\nu}_{\mu} + \Delta\omega^{\nu}_{\mu} = \delta^{\nu}_{\mu} + \Delta\omega\left(I_n\right)^{\nu}_{\mu} \tag{4.54}$$

$\Delta\omega$ is an infinitesimal quantity, and I_n is a 4 x 4 matrix corresponding to a specific Lorentz transformation.

For a transformation to S′ moving along the x-axis with an infinitesimal velocity $\Delta\omega = \Delta\beta$, we have

$$I_\mu^\nu = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad I^{\nu\mu} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.55}$$

Similarly, for a rotation through an angle $\Delta\omega$ around the z-axis, the correspond I_μ^ν is

$$I_\mu^\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad I^{\nu\mu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.56}$$

In order to satisfy the relation for Lorentz transformation

$$\Lambda_\mu^\nu \Lambda_\sigma^\mu = \delta_\sigma^\nu \tag{4.57}$$

the infinitesimal transformation, Equation 4.54, satisfies

$$I^{\mu\nu} = -I^{\nu\mu} \tag{4.58}$$

as can be readily verified for Equations 4.55 and 4.56.

There are 6 independent I_μ^ν, 3 for boosts and 3 for rotations.

Expand S in powers of $\Delta\omega$*:*

$$S = 1 - \frac{i}{4}\sigma_{\mu\nu}\Delta\omega^{\mu\nu} \tag{4.59}$$

$\sigma_{\mu\nu}$ is the generator for the corresponding infinitesimal Lorentz transformation.

Inserting Equation 4.59 and Equation 4.54 into Equation 4.53, we have

$$2i\left[g_{\alpha}^{\nu}\ \gamma_{\beta} - g_{\beta}^{\nu}\ \gamma_{\alpha} \right] = \left[\gamma^{\nu},\ \sigma_{\alpha\beta} \right] \tag{4.60}$$

which is satisfied by

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_{\mu},\ \gamma_{\nu} \right] \tag{4.61}$$

The anti-symmetric tensor $\sigma_{\mu\nu}$ has the following non-vanishing elements

$$\sigma_{oi} = -\sigma_{io} = -i\alpha_{i} \tag{4.62}$$

for boost and

$$\sigma_{ij} = -\sigma_{ji} = \begin{bmatrix} \sigma_{k} & 0 \\ 0 & \sigma_{k} \end{bmatrix} \tag{4.63}$$

for rotation *ijk* cyclic.

We are ready to construct a finite spinor transformation *S* from successive infinitesimal transformation

$$\begin{aligned} S &= \lim_{N\to\infty}\left(1 - \frac{i}{4}\frac{\omega}{N}\sigma_{\mu\nu}I_{n}^{\mu\nu} \right)^{N} \\ &= \exp\left(-\frac{i}{4}\omega\sigma_{\mu\nu}I_{n}^{\mu\nu} \right) \end{aligned} \tag{4.64}$$

To prove Equation 4.60

$$S = 1 - \frac{i}{4}\sigma_{\alpha\beta}\Delta\omega^{\alpha\beta}$$

$$S^{+} = 1 + \frac{i}{4}\sigma_{\alpha\beta}\Delta\omega^{\alpha\beta}$$

$$\Lambda_{\mu}^{\nu} = \delta_{\mu}^{\nu} + \Delta\omega_{\mu}^{\nu}$$

Since $$\Lambda_{\mu}^{\nu}\ \gamma^{\mu} = S^{-1}\ \gamma^{\nu}\ S$$

We have $$\left(\delta_{\mu}^{\nu} + \Delta\omega_{\mu}^{\nu} \right)\gamma^{\mu} = \left(1 + \frac{i}{4}\sigma_{\alpha\beta}\Delta\omega^{\alpha\beta} \right)\gamma^{\nu}\left(1 - \frac{i}{4}\sigma_{\alpha\beta}\Delta\omega^{\alpha\beta} \right)$$

or $$\gamma^{\nu} + \Delta\omega^{\nu}_{\mu}\ \gamma^{\mu} = \gamma^{\nu} - \frac{i}{4}\gamma^{\nu}\sigma_{\alpha\beta}\Delta\omega^{\alpha\beta} + \frac{i}{4}\sigma_{\alpha\beta}\Delta\omega^{\alpha\beta}\gamma^{\nu}$$

(ignoring $(\Delta\omega)^2$ term)

Therefore $$\Delta\omega^{\nu}_{\mu}\ \gamma^{\mu} = -\frac{i}{4}\Delta\omega^{\alpha\beta}\left[\gamma^{\nu},\sigma_{\alpha\beta}\right] \qquad \text{(a)}$$

but $$\Delta\omega^{\nu}_{\mu}\ \gamma^{\mu} = \Delta\omega^{\nu\mu}\gamma_{\mu} = \frac{1}{2}\left[\Delta\omega^{\nu\beta}\gamma_{\beta} + \Delta\omega^{\nu\alpha}\gamma_{\alpha}\right]$$
$$= \frac{1}{2}\left[\Delta\omega^{\nu\beta}\gamma_{\beta} - \Delta\omega^{\alpha\nu}\gamma_{\alpha}\right]$$
$$= \frac{1}{2}\left[g^{\nu}_{\alpha}\ \gamma_{\beta} - g^{\nu}_{\beta}\ \gamma_{\alpha}\right] = \left[\gamma^{\nu},\ \sigma_{\alpha\beta}\right] \qquad \text{(b)}$$

Inserting (b) into (a), we obtain

$$2i\left[g^{\nu}_{\alpha}\ \gamma_{\beta} - g^{\nu}_{\beta}\ \gamma_{\alpha}\right] = \left[\gamma^{\nu},\ \sigma_{\alpha\beta}\right]$$

For a Lorentz boost along the *x*-axis,

$$\psi'(x') = \exp\left[\left(-\frac{i}{2}\right)\omega\ \sigma_{01}\right]\psi(x)$$
$$= \exp\left(-\frac{\omega}{2}\alpha_1\right)\psi(x) \qquad (4.65)$$

where tank $\omega = \beta$, cosh $\omega = \frac{1}{\sqrt{1-\beta^2}} = \gamma$, sinh $\omega = \beta\gamma$

Similarly, for a rotation around the *z*-axis of an angle ω:

$$\psi'(x') = \exp\left[\frac{i}{2}\omega\ \sigma^{12}\right]\psi(x) \qquad (4.66)$$

where $$\sigma^{12} = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix}$$

Equation 4.66 is similar to the rotation of two-component Pauli spinor

$$\varphi'(x') = e^{(i/2)\vec{\omega}\cdot\vec{\sigma}}\varphi(x) \tag{4.67}$$

Are the spinor transformation *S* unitary?

For spatial rotation, $S = S_{Rot}$, *S* is unitary, since

$$S_{Rot}^{+} = e^{-i/2\,\omega(\sigma^{ij})^{+}} = e^{-i/2\,\omega(\sigma^{ij})} = S_{Rot}^{-1} \tag{4.68}$$

but for Lorentz boost

$$S_{Lor}^{+} = e^{(-\omega/2)\alpha_i^{+}} = e^{(-\omega/2)\alpha_i} = S_{Lor} \neq S_{Lor}^{-1} \tag{4.69}$$

Nevertheless, both S_{Rot} and S_{Lor} have the property

$$S^{-1} = \gamma_o\, S^{+}\, \gamma_o \tag{4.70}$$

which can be verified by expanding S_{Rot}, S_{Lor} in power series.

Here is proof for $\beta = \tanh\omega$

Consider a Lorentz boost along the *x*-axis

$$I = I_{\mu}^{\nu} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$I^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad I^3 = I$$

$$\chi'^{\nu} = \lim_{N\to\infty}\left(1 + \frac{\omega}{N} I\right)_{\mu}^{\nu} x^{\mu}$$

$$= \left(e^{\omega I}\right)^{\nu}_{\mu} x^{\mu}$$

$$= \left(\cosh \omega I + \sinh \omega I\right)^{\nu}_{\mu} x^{\mu}$$

$$= \left(1 - I^2 + I^2 \cosh \omega + I \sinh \omega\right)^{\nu}_{\mu} x^{\mu}$$

$$x' = \begin{bmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$

$$t' = (\cosh \omega)\, t - (\sinh \omega)\, x$$
$$x' = -(\sinh \omega)\, t + (\cosh \omega)\, x$$

We also have

$$t' = \gamma\, (t - \beta x)$$
$$x' = \gamma\, (-\beta t + x)$$

Therefore

$$\gamma = \cosh \omega \qquad \gamma\beta = \sinh \omega \qquad \beta = \tanh \omega$$

The fact that S_{Lor} is not unitary (Equation 4.69) should not be too surprising. This simply reflects the situation that $\psi^{+}\psi$ is not a conserved quantity in a Lorentz boost.

We now consider space inversion:

$$\Lambda^{\nu}_{\mu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

This discrete transformation cannot be constructed out of infinitesimal transformation. Rather, we rely on the definition of S (Equation 4.53) to find the parity transformation S_p for the Dirac spinor ψ.

Equation 4.53 and Equation 4.71 imply

$$S_p^{-1} \gamma^o S_p = \gamma^o$$
$$S_p^{-1} \gamma^i S_p = -\gamma^i \qquad (4.72)$$

This can be satisfied with

$$S_p = \gamma^o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Note that S_p also satisfies Equation 4.70 since

$$S_p^{-1} = \gamma^o S_p^+ \gamma_o$$

For a Dirac particle at rest, Equation 4.73 shows that U_1 and U_2 have positive parity while U_3 and U_4 have negative parity (Equation 4.46)!

Having established the properties of S, we can show that $\bar{\psi}\psi$ transforms like a scalar, while $\bar{\psi}\gamma^\mu\psi$ transforms like a 4-vector under Lorentz transformation.

$$\begin{aligned} &\bar{\psi}'(x') = \psi'(x')^+ \gamma_o = \left(S\psi(x)\right)^+ \gamma_o = \psi(x)^+ S^+\gamma_o \\ &= \psi(x)^+ \gamma_o S^{-1} = \bar{\psi}(x) S^{-1} \end{aligned} \tag{4.74}$$

Therefore

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x) S^{-1} S\psi(x) = \bar{\psi}(x)\psi(x) \tag{4.75}$$

$$\Rightarrow \quad \bar{\psi}\psi \text{ is a scalar}$$

$$\begin{aligned} &\bar{\psi}'(x')\gamma^\mu\psi'(x') = \bar{\psi}(x) S^{-1}\gamma^\mu S\psi(x) \\ &= \bar{\psi}(x)\Lambda^\mu_\nu\gamma^\nu\psi(x) \\ &= \Lambda^\mu_\nu\left(\bar{\psi}(x)\gamma^\nu\psi(x)\right) \end{aligned}$$

$$\Rightarrow \quad \bar{\psi}\gamma^\mu\psi \text{ is a 4-vector}$$

To obtain the plane-wave solution to the Dirac equation for a particle moving along the x-axis with velocity β, we boost along $-x$ with a velocity of $-\beta$.

$$S = e^{-\left(i/2\right)\omega\sigma_{01}}; \qquad \sigma_{01} = -i\alpha_1$$
$$= e^{-\omega/2\,\alpha_1} = \cosh\frac{\omega}{2} - \alpha_1 \sinh\frac{\omega}{2} \tag{4.76}$$

where tanh ω = -β

The spinors $U^\nu(p)$ are

$$U^\nu(p) = \left[\cosh\left(\frac{\omega}{2}\right) - \alpha_1 \sinh\left(\frac{\omega}{2}\right)\right] U^\nu(0)$$

$$= \cosh\left(\frac{\omega}{2}\right) \begin{bmatrix} 1 & 0 & 0 & -\tanh\frac{\omega}{2} \\ 0 & 1 & -\tanh\frac{\omega}{2} & 0 \\ 0 & -\tanh\frac{\omega}{2} & 1 & 0 \\ -\tanh\frac{\omega}{2} & 0 & 0 & 1 \end{bmatrix} U^\nu(0) \tag{4.77}$$

where

$$\tanh\left(\frac{\omega}{2}\right) = \frac{\tanh\omega}{1+\sqrt{1-\tanh^2\omega}} = \frac{-\beta}{1+\left(1-\beta^2\right)^{1/2}}$$
$$= \frac{-\beta}{1+1/\gamma} = \frac{-\beta\gamma}{1+\gamma} = \frac{-P}{\mathrm{E}+m} \tag{4.78}$$

and

$$\cosh\left(\frac{\omega}{2}\right) = \frac{1}{\left(1-\tanh^2\left(\frac{\omega}{2}\right)\right)^{1/2}} = \frac{1}{\left(\frac{2}{1+\gamma}\right)^{1/2}} = \sqrt{\frac{m+\mathrm{E}}{2m}}$$

using Equation 4.78, Equation 4.77 becomes

$$U^\nu(p) = \sqrt{\frac{m+\mathrm{E}}{2m}} \begin{bmatrix} 1 & 0 & 0 & Px/\mathrm{E}+m \\ 0 & 1 & Px/\mathrm{E}+m & 0 \\ 0 & Px/\mathrm{E}+m & 1 & 0 \\ Px/\mathrm{E}+m & 0 & 0 & 1 \end{bmatrix} U^\nu(0) \tag{4.79}$$

For a boost along an arbitrary direction

$$I^{\mu}_{\nu} = \begin{bmatrix} 0 & -Px/|p| & -Py/|p| & -Pz/|p| \\ -Px/|p| & 0 & 0 & 0 \\ -Py/|p| & 0 & 0 & 0 \\ -Pz/|p| & 0 & 0 & 0 \end{bmatrix} \tag{4.80}$$

and

$$S = e^{-\omega/2\,\vec{\alpha}\bullet\hat{P}} \tag{4.81}$$

$$U^{\nu}(p) = \sqrt{\frac{m+\mathrm{E}}{2m}} \begin{bmatrix} 1 & 0 & \frac{Pz}{\mathrm{E}+m} & \frac{Px-iPy}{\mathrm{E}+m} \\ 0 & 1 & \frac{Px+iPy}{\mathrm{E}+m} & \frac{-Pz}{\mathrm{E}+m} \\ \frac{Pz}{\mathrm{E}+m} & \frac{Px-iPy}{E+m} & 1 & 0 \\ \frac{Px+iPy}{\mathrm{E}+m} & \frac{-Pz}{\mathrm{E}+m} & 0 & 1 \end{bmatrix} U^{\nu}(0) \tag{4.82}$$

The four solutions of the spinors are

$$U^1(p)=\sqrt{\frac{m+\mathrm{E}}{2m}}\begin{bmatrix}1\\0\\\frac{Pz}{\mathrm{E}+m}\\\frac{Px+iPy}{\mathrm{E}+m}\end{bmatrix};\quad U^2(p)=\sqrt{\frac{m+\mathrm{E}}{2m}}\begin{bmatrix}1\\0\\\frac{Px-iPy}{\mathrm{E}+m}\\\frac{-Pz}{\mathrm{E}+m}\end{bmatrix}$$

$$U^3(p)=\sqrt{\frac{m+\mathrm{E}}{2m}}\begin{bmatrix}\frac{Pz}{\mathrm{E}+m}\\\frac{Px+iPy}{\mathrm{E}+m}\\1\\0\end{bmatrix};\quad U^4(p)=\sqrt{\frac{m+\mathrm{E}}{2m}}\begin{bmatrix}\frac{Px-iPy}{\mathrm{E}+m}\\\frac{-Pz}{\mathrm{E}+m}\\0\\1\end{bmatrix}\qquad(4.83)$$

Note that Equation 4.83 is obtained by using a normalization:

$$U^{(1)}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix};\quad U^{(2)}=\begin{bmatrix}0\\1\\0\\0\end{bmatrix};\quad U^{(3)}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix};\quad U^{(4)}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\qquad(4.84)$$

We can also solve the Dirac equation directly:

$$\mathrm{E}U=\left(\vec{\alpha}\bullet\vec{P}+\beta m\right)U=\begin{pmatrix}m & \vec{\sigma}\bullet\vec{P}\\ \vec{\sigma}\bullet\vec{P} & -m\end{pmatrix}U\qquad(4.85)$$

$U=\begin{pmatrix}U_A\\U_B\end{pmatrix}$, where U_A, U_B are two-component wave functions

$$\mathrm{E}\begin{pmatrix}U_A\\U_B\end{pmatrix}=\begin{pmatrix}m & \vec{\sigma}\bullet\vec{P}\\ \vec{\sigma}\bullet\vec{P} & -m\end{pmatrix}\begin{pmatrix}U_A\\U_B\end{pmatrix}$$

We obtain the coupled equations

$$\vec{\sigma}\bullet\vec{P}U_B = (\mathrm{E} - m)U_A \quad \text{and} \quad \vec{\sigma}\bullet\vec{P}U_A = (\mathrm{E} + m)U_B \tag{4.86}$$

A possible solution to Equation 4.86 is

$$\text{For E} > 0,\ U^{(S)} = N\begin{pmatrix} x^{(S)} \\ \dfrac{\vec{\sigma}\bullet\vec{P}}{E+m}x^{(S)} \end{pmatrix}, \qquad x^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad x^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S = 1,\ 2$$

$$\text{For E} < 0,\ U^{(S+2)} = N\begin{pmatrix} \dfrac{\vec{\sigma}\bullet\vec{P}}{E-m}x^{(S)} \\ x^{(S)} \end{pmatrix}$$

Explicitly:

$$U^{(1)}(p) = N\begin{pmatrix} 1 \\ 0 \\ \dfrac{Pz}{E+m} \\ \dfrac{Px+iPy}{E+m} \end{pmatrix}; \quad U^{(2)}(p) = N\begin{pmatrix} 0 \\ 1 \\ \dfrac{Px-iPy}{E+m} \\ \dfrac{-Pz}{E+m} \end{pmatrix}; \quad \mathrm{E} > 0$$

$$U^{(3)}(p) = N\begin{pmatrix} \dfrac{Pz}{E-m} \\ \dfrac{Px+iPy}{E-m} \\ 1 \\ 0 \end{pmatrix}; \quad U^{(4)}(p) = N\begin{pmatrix} \dfrac{Px-iPy}{E-m} \\ \dfrac{-Pz}{E-m} \\ 0 \\ 1 \end{pmatrix}; \quad \mathrm{E} < 0 \tag{4.87}$$

Note that $U^{(3)}(p)$, $U^{(4)}(p)$ from Equation 4.87 are different from those obtained by the boost method (Equation 4.83). Apart from a difference in the normalization factor N,

$$U^{(3,4)}_{\text{boost}}(p) = U^{(3,4)}_{\text{direct}}(-p)\text{, where } p = (E, \vec{p})$$

Note that E > 0 for $U^{(3,4)}_{\text{boost}}(p)$, while E < 0 for $U^{(3,4)}_{\text{direct}}(p)$.

In order to interpret the E < 0 solutions for $U^{(3,4)(}$p), we examine the charge-conjugation transformation on the Dirac equation.

The Dirac equation for an electron in an EM field can be obtained with the gauge substitution

$$\partial^\mu \to D^\mu \equiv \partial^\mu + iqA^\mu \tag{4.89}$$

For an electron withy charge q = -e (e > 0), we have

$$i\partial_\mu \to i\partial_\mu + eA_\mu \tag{4.90}$$

The Dirac equation for an electron becomes

$$\left[\gamma^\mu\left(i\partial_\mu + eA_\mu\right) - m\right]\psi = 0 \tag{4.91}$$

The Dirac equation for a positron, ψ_c, would be

$$\left[\gamma^\mu\left(i\partial_\mu - eA_\mu\right) - m\right]\psi_c = 0 \tag{4.92}$$

Our task is to find the transformation linking ψ to ψ_c. To change the relative sign between the ∂_μ and A_μ terms in Equation 4.91, one can take a complex conjugate of Equation 4.91

$$\left[\left(\gamma^\mu\right)^*\left(-i\partial_\mu + eA_\mu\right) - m\right]\psi^* = 0 \tag{4.93}$$

To find a 4 x 4 matrix S_c, which is conventionally written as $S_c = c\gamma^0$, that satisfies

$$S_c\,\psi^* = \psi_c \tag{4.94}$$

We insert $S_c^+ S_c$ term in front of ψ^* in Equation 4.93 and multiply S_c from the left:

$$\left[S_c\left(\gamma^\mu\right)^* S_c^{-1}\left(-i\partial_n + eA_\mu\right) - m\right]\left(S_c\psi^*\right) = 0 \tag{4.95}$$

From Equations 4.94 and 4.95, it follows that S_c satisfies the following equation:

$$S_c\left(\gamma^\mu\right)^* S_c^{-1} = -\gamma^\mu \tag{4.96}$$

Now, the Pauli-Dirac representation of γ^μ gives

$$\begin{aligned} \left(\gamma^\mu\right)^* &= \gamma^\mu \quad \text{for } \mu = 0,\ 1,\ 3 \\ \left(\gamma^\mu\right)^* &= -\gamma^\mu \quad \text{for } \mu = 2 \end{aligned} \tag{4.97}$$

Therefore, Equation 4.96 implies that S_c commutes with γ^2 and anticommutes with γ^0, γ^1, γ^3. This can be satisfied if S_c is proportional to γ^2:

$$S_c = i\gamma^2 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} \tag{4.98}$$

The factor i is chosen by convention.

To obtain the positron wave function, one takes a complex conjugate of the electron wave function, followed by a multiplication of $i\gamma^2$:

$$\psi_c = i\gamma^2\psi^* \tag{4.99}$$

We can apply Equation 4.99 to $\psi^{(1)}(p)$:

$$\psi^{(1)}(p) = N\begin{pmatrix} 1 \\ 0 \\ \dfrac{Pz}{\mathrm{E}+m} \\ \dfrac{Px+iPy}{\mathrm{E}+m} \end{pmatrix} e^{-iP\bullet x}, \qquad P^o = \mathrm{E} > 0 \tag{4.100}$$

$$i\gamma^2\psi^{(1)*}(p) = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} N\begin{pmatrix} 1 \\ 0 \\ \dfrac{Pz}{\mathrm{E}+m} \\ \dfrac{Px-iPy}{\mathrm{E}+m} \end{pmatrix} e^{iP\bullet x}$$

$$= N\begin{pmatrix} \dfrac{Px-iPy}{\mathrm{E}+m} \\ \dfrac{-Pz}{\mathrm{E}+m} \\ 0 \\ 1 \end{pmatrix} e^{iP\bullet x}; \qquad P^o = \mathrm{E} > 0 \tag{4.101}$$

We can also write $\psi^{(4)}(p')$ as

$$\psi^{(4)}(p') = N\begin{pmatrix} \dfrac{Px'-iPy'}{\mathrm{E}\text{-}m} \\ \dfrac{-Pz'}{\mathrm{E}\text{-}m} \\ 0 \\ 1 \end{pmatrix} e^{-iP'\bullet x} \quad \text{where} \quad (P')^0 = E' < 0 \tag{4.102}$$

Substituting $p' = -p$ into Equation 4.102, we obtain

$$\psi^{(4)}(-p) = N \begin{pmatrix} \dfrac{-Px + iPy}{-\mathrm{E} - m} \\ \dfrac{Pz}{-\mathrm{E} - m} \\ 0 \\ 1 \end{pmatrix} e^{iP \bullet x}; \quad P^0 = \mathrm{E} > 0 \tag{4.103}$$

Equation 4.103 is identical to Equation 4.101.

$$\begin{aligned} &\text{Hence,} \quad \psi_c^{(1)} = i\gamma^2 \psi^{(1)*}(p) = \psi^{(4)}(-p) \\ &\text{Similarly,} \quad \psi_c^{(2)} = i\gamma^2 \psi^{(2)*}(p) = -\psi^{(3)}(-p) \end{aligned} \tag{4.104}$$

It also follows

$$\begin{aligned} U^{(4)}(-p) &= v^{(1)}(p) \\ -U^{(3)}(-p) &= v^{(2)}(p) \end{aligned} \tag{4.105}$$

where $v^{(1)}$, $v^{(2)}$ are the spinors for positron wave functions.

Note that the definition of $v^{(2)}$ (p) in Halzen and Martin differs from Equation 4.105 by a minus sign ($U^3(-p) = v^{(2)}(p)$ in H & M).

It is instructive to consider time-reversal operation on the Dirac equation. In a procedure analogous to that for the charge-conjugation operation, we can write down the Dirac equation for a time-reversed electron state, ψ_t, as

$$\left(-i\gamma^0 \frac{\partial}{\partial t} + i\gamma^1 \frac{\partial}{\partial x} + i\gamma^2 \frac{\partial}{\partial y} + i\gamma^3 \frac{\partial}{\partial z} - m \right) \psi_t = 0 \tag{4.106}$$

and the ordinary Dirac equation as

$$\left(i\gamma^0 \frac{\partial}{\partial t} + i\gamma^1 \frac{\partial}{\partial x} + i\gamma^2 \frac{\partial}{\partial y} + i\gamma^3 \frac{\partial}{\partial z} - m \right) \psi = 0 \tag{4.107}$$

Note that Equation 4.106 is obtained by $t \to -t$ transformation applied to the Dirac equation.

Taking the complex conjugate of Equation 4.107 gives

$$\left(-i\left(\gamma^0\right)^*\frac{\partial}{\partial t}-i\left(\gamma^1\right)^*\frac{\partial}{\partial x}-i\left(\gamma^2\right)^*\frac{\partial}{\partial y}-i\left(\gamma^3\right)^*\frac{\partial}{\partial z}-m\right)\psi^*=0 \qquad (4.108)$$

To find a 4 x 4 matrix which transforms ψ to ψ_t

$$S_t\,\psi^* = \psi_t \qquad (4.109)$$

When we multiply $S_t^{-1}S_t$ in front of ψ* in Equation 4.108 followed by multiplying S_t from the left, we get

$$\left\{\left[S_t\left(-i\left(\gamma^0\right)^*\frac{\partial}{\partial t}-i\left(\gamma^1\right)^*\frac{\partial}{\partial x}-i\left(\gamma^2\right)^*\frac{\partial}{\partial y}-i\left(\gamma^3\right)^*\frac{\partial}{\partial z}\right)S_t^{-1}\right]-m\right\}S_t\psi^*=0 \qquad (4.110)$$

Equation 4.109 and 4.110 imply that S_t should have the following properties:

$$\begin{aligned} S_t\left(\gamma^0\right)^* S_t^{-1} &= \gamma^0 \\ S_t\left(\gamma^i\right)^* S_t^{-1} &= -\gamma^i \qquad (i=1\rightarrow 3) \end{aligned} \qquad (4.111)$$

Since $(\gamma^0)^* = \gamma^0$, $(\gamma^1)^* = \gamma^1$, $(\gamma^2)^* = -\gamma^2$, $(\gamma^3)^* = \gamma^3$, Equation 4.111 implies

S_t commutes with γ^0, γ^2 and anticommutes with γ^1, γ^3. S_t is therefore proportional to γ^1, γ^3 and, by convention, one chooses

$$\begin{aligned} S_t &= i\gamma^1\gamma^3 \\ &= \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} = -i\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned} \qquad (4.112)$$

Applying $S_t\,\psi^*(p)$ for $\psi^{(1)}$, one obtains

$$S_t\psi^{(1)*}(p) = -i\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} N \begin{pmatrix} 1 \\ 0 \\ \frac{Pz}{E+m} \\ \frac{Px - iPy}{E+m} \end{pmatrix} e^{iP\bullet x}$$
$$= -iN\begin{pmatrix} 0 \\ 1 \\ \frac{-Px - iPy}{E+m} \\ \frac{Pz}{E+m} \end{pmatrix} e^{iP\bullet x} \qquad (4.113)$$

Hence, the time-reversal operation transforms $\psi^{(1)}(\vec{p},t)$ to $\psi^{(2)}(-\vec{p},-t)$, as one would expect.

The combined operation of CPT can be obtained too.

Since

$$P\psi = \gamma^0\psi$$
$$C\psi = i\gamma^2\psi*$$
$$T\psi = i\gamma^1\gamma^3\psi*$$

we have

$$\begin{aligned} \psi_{CPT} &= CPT\psi = CP\left(i\gamma^1\gamma^3\psi^*\right) \\ &= C\left(\gamma^0 i\gamma^1\gamma^3\psi^*\right) \\ &= -i\gamma^2\gamma^0 i\gamma^1\gamma^3\psi \\ &= \gamma^0\gamma^1\gamma^2\gamma^3\psi \\ &= -i\gamma^5\psi \end{aligned}$$

For Klein-Gordon equation

$$\left(\partial^\mu\partial_\mu + m^2\right)\phi(x) = 0$$

In the presence of an EM field:

$$i\partial^\mu \rightarrow i\partial^\mu - eA^\mu$$

and the K-G equation becomes

$$\left(i\partial^\mu - eA^\mu\right)\left(i\partial_\mu - eA_\mu\right)\phi(x) = m^2\phi(x)$$

or

$$\left[\Box^2 + m^2 + ie\left(\partial^\mu A_\mu + A^\mu \partial_\mu\right) - e^2 A^2\right]\phi(x) = 0$$

It is straight forward to show that

$$\begin{aligned} \text{P:}\quad & \phi'_P(x') = \phi(x) \\ \text{C:}\quad & \phi'_C(x') = \phi^*(x) \\ \text{T:}\quad & \phi'_T(x') = \phi^*(x) \\ \text{CPT:}\quad & \phi'_{CPT}(x') = \phi(x) \end{aligned}$$

One can also work out the non-relativistic limit of the Dirac equation. From Equation 4.86, the following coupled equations can be obtained:

$$\begin{aligned} \left(\vec{\sigma}\bullet\vec{P}\right)\psi_B &= \left(\mathrm{E} - m\right)\psi_A \\ \left(\vec{\sigma}\bullet\vec{P}\right)\psi_A &= \left(\mathrm{E} + m\right)\psi_B \end{aligned} \tag{4.114}$$

where

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

Making the substitution $P \rightarrow P + eA = \pi$, Equation 4.114 becomes

$$\begin{aligned} \left(\vec{\sigma}\bullet\vec{\pi}\right)\psi_B &= \left(E + eA^o - m\right)\psi_A \\ \left(\vec{\sigma}\bullet\vec{\pi}\right)\psi_A &= \left(E + eA^o + m\right)\psi_B \end{aligned} \tag{4.115}$$

In the non-relativistic limit $\mathrm{E} \simeq m$ and $eA^{\mathrm{o}} << m$.
Therefore

$$\psi_B \simeq \frac{\vec{\sigma}\bullet\vec{\pi}}{2m}\psi_A \tag{4.116}$$

Using Equation 4.115, Equation 4.116 gives

$$\frac{\left(\vec{\sigma}\bullet\vec{\pi}\right)^2}{2m}\psi_A = \left(\mathrm{E}_{NR} + eA^o\right)\psi_A \tag{4.117}$$

where $\mathrm{E}_{NR} > \mathrm{E} - m$

$$\begin{aligned}\left(\vec{\sigma}\bullet\vec{\pi}\right)^2 &= \vec{\pi}^2 + i\vec{\sigma}\bullet\left(\vec{\pi}\times\vec{\pi}\right) = \left(\vec{P}+e\vec{A}\right)^2 + i\vec{\sigma}\bullet\left(\vec{P}+e\vec{A}\right)\times\left(\vec{P}+e\vec{A}\right)\\ &= \left(\vec{P}+e\vec{A}\right)^2 + e\left(\vec{\nabla}\times\vec{A}\right)\bullet\vec{\sigma}\end{aligned} \tag{4.118}$$

Note that $\vec{P}\times\left(\vec{A}\psi\right) + \vec{A}\times\left(\vec{P}\psi\right) = -i\left(\vec{\nabla}\times\vec{A}\right)\psi$

Equations 4.117 and 4.118 imply

$$\left[\frac{1}{2m}\left(\vec{P}+e\vec{A}\right)^2 + \frac{e}{2m}\vec{\sigma}\bullet\vec{B} - eA^o\right]\psi_A = \mathrm{E}_{NR}\psi_A \tag{4.119}$$

which is the Pauli equation.

For a uniform $\vec{B}$ field, $\vec{A} = \frac{1}{2}\vec{B}\times\vec{\gamma}$

$$\left(\vec{P}+e\vec{A}\right)^2 \cong P^2 + 2e\vec{P}\bullet\frac{1}{2}\left(\vec{B}\times\vec{\gamma}\right) = P^2 + e\vec{B}\bullet\left(\vec{\gamma}\times\vec{P}\right) = P^2 + e\vec{B}\bullet\vec{L} \tag{4.120}$$

Using Equation 4.120, Equation 4.119 becomes

$$\left[\frac{P^2}{2m} + \frac{e}{2m}\left(\vec{L}+2\vec{S}\right)\bullet\vec{B} - eA^o\right]\psi_A = \mathrm{E}_{NR}\psi_A \tag{4.121}$$

The Dirac equation leads to $g_s = 2$ for spin-½ particle.

However, $g_s = 2$ is not a unique feature of Dirac equation. To illustrate this point, one can start with a non-relativistic expression

$$H = P^2/2m = \left(\vec{\sigma}\bullet\vec{P}\right)/2m \tag{4.122}$$

Making the $\vec{P} \to \vec{P} + e\vec{A}$ substitution, Equation 4.122 becomes

$$\begin{aligned} H &= \left[\vec{\sigma} \bullet \left(\vec{P} + e\vec{A} \right) \right]^2 \Big/ 2m \\ &= \frac{1}{2m} \left[\left(\vec{P} + e\vec{A} \right)^2 + e\vec{\sigma} \bullet \vec{B} \right] \\ &= \frac{1}{2m} \left[\left(\vec{P} + e\vec{A} \right)^2 \right] + \frac{e}{2m} \left(2\vec{S} \bullet \vec{B} \right) \end{aligned} \tag{4.123}$$

Hence, $g_s = 2$ also follows from a non-relativistic approach.

The spin operator is defined as

$$\sum\nolimits_k = \sigma_{ij} = \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix} \quad \left(i,\ j,\ k = 1,\ 2,\ 3 \ \text{cyclic} \right) \tag{4.124}$$

It can be shown that the Dirac Hamiltonian $H = \vec{\alpha} \bullet \vec{P} + \beta m$ does not commute with Σ, namely

$$\left[H, \vec{\Sigma} \right] = 2i \left(\vec{\alpha} \times \vec{P} \right) \neq 0 \tag{4.125}$$

It does not commute with the orbital angular momentum $\vec{L}$ either

$$\left[H, \vec{L} \right] = -i \left(\vec{\alpha} \times \vec{P} \right) \neq 0 \tag{4.126}$$

However, Equations 4.125 and 4.126 show that H commutes with the total angular momentum $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$

$$\left[H,\ \vec{J} \right] = \left[H,\ \vec{L} \right] + \left[H,\ \frac{1}{2}\vec{\Sigma} \right] = 0 \tag{4.127}$$

Although spin itself is not conserved, the spin projection along the momentum vector is conserved. This spin projection is also called helicity and is defined as

$$\frac{1}{2}\vec{\Sigma}\bullet\vec{P} = \frac{1}{2}\begin{bmatrix} \vec{\sigma}\bullet\hat{P} & 0 \\ 0 & \vec{\sigma}\bullet\hat{P} \end{bmatrix} \tag{4.128}$$

One can readily show that $\left[H,\ \frac{1}{2}\vec{\Sigma}\bullet\hat{P}\right]=0$. The reason why $\frac{1}{2}\vec{\Sigma}\bullet\hat{P}$ is conserved can be understood if one recognizes that there is no orbital angular momentum along the direction of momentum. Hence the particle's spin is the same as its total angular momentum in this case, and it is conserved.

Note that $\Sigma^k = \gamma^5\gamma^0\gamma^k \qquad (k=1,\ 2,\ 3)$

The Dirac spinors given in Equation 4.87 are not eigenstates of the helicity operator:

$$\left(\frac{1}{2}\vec{\Sigma}\bullet\hat{P}\right)U^{(1)}(P) = \frac{1}{2|P|}\begin{pmatrix} Pz & Px-iPy & 0 & 0 \\ Px+iPy & -Pz & 0 & 0 \\ 0 & 0 & Pz & Px-iPy \\ 0 & 0 & Px+iPy & -Pz \end{pmatrix} N \begin{pmatrix} 1 \\ 0 \\ Pz/E+m \\ Px+iPy/E+m \end{pmatrix}$$

$$= \frac{N}{2|P|}\begin{pmatrix} Pz \\ Px+iPy \\ P^2/E+m \\ 0 \end{pmatrix} \tag{4.129}$$

Equation 4.129 shows that in general, $U^{(1)}(P)$ is not an eigenstate of $\frac{1}{2}\vec{\Sigma}\bullet\hat{P}$. In the special case when $Px = Py = 0$, $U^{(1)}(P)$ is an eigenstate of $\frac{1}{2}\vec{\Sigma}\bullet\hat{P}$ with an eigenvalue of +½.

It is possible to construct basis states which are eigenstates of the helicity operator. It can easily be shown that the following $U^{(\pm)}$ are hilicity eigenstates with eigenvalues of +½, -½, respectively.

$$U^{(\pm)} = N \begin{pmatrix} Pz \pm |P| \\ Px + iPy \\ \left(Pz \pm |P|\right)\frac{|P|}{E+m} \\ \left(Px + iPy\right)\frac{|P|}{E+m} \end{pmatrix} \tag{4.130}$$

We summarize here some of the pertinent properties of the Dirac spinors *U* and *v*:

First, *U* and *v* satisfy the following equations:

$$\begin{aligned} &\left(\not{P} - m\right)U = 0, \quad \left(\not{P} + m\right)v = 0 \\ &\bar{U}\left(\not{P} - m\right) = 0, \quad \bar{v}\left(\not{P} + m\right) = 0 \end{aligned} \tag{4.131}$$

The orthogonality relations are

$$\begin{aligned} &U^{(r)^+}U^{(s)} = 2E\ \delta rs, \quad v^{(r)^+}v^{(s)} = 2E\ \delta rs \\ &\bar{U}^{(s)}U^{(s)} = 2m, \quad \bar{v}^{(s)}v^{(s)} = -2m \end{aligned} \tag{4.132}$$

Note that different normalizations have been adopted in the literature. For example

$$\bar{U}^{(s)}U^{(s)} = 1, \quad \bar{v}^{(s)}v^{(s)} = 1 \tag{4.133}$$

have been chosen by Bjorken and Drell's textbook.

The completeness relations are

$$\begin{aligned} &\sum_{s=1,2} U^{(s)}(P)\bar{U}^{(s)}(P) = \not{P} + m \\ &\sum_{s=1,2} v^{(s)}(P)\bar{v}^{(s)}(P) = \not{P} - m \end{aligned} \tag{4.134}$$

Why are these relations called comleteness relations?

One can show easily that

$$\sum_{s=1,2} U^{(s)}(P)\bar{U}^{(s)}(P) - v^{(s)}(P)\bar{v}^{(s)}(P) = 2m \tag{4.135}$$

Using the normalization scheme of Equation 4.133, one obtains

$$\sum_{s=1,2} U^{(s)}(P)\bar{U}^{(s)}(P) - v^{(s)}(P)\bar{v}^{(s)}(P) = 1$$

Equations 4.134 are very important for evaluating cross sections later on.

Explicitly

$$U^{(1)} = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ Pz/E+m \\ Px+iPy/E+m \end{pmatrix}; \quad U^{(2)} = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ Px-iPy/E+m \\ -Pz/E+m \end{pmatrix}$$

$$v^{(1)} = \sqrt{E+m}\begin{pmatrix} Px-iPy/E+m \\ -Pz/E+m \\ 0 \\ 1 \end{pmatrix}; \quad v^{(2)} = \sqrt{E+m}\begin{pmatrix} Pz/E+m \\ Px+iPy/E+m \\ 1 \\ 0 \end{pmatrix}$$

In addition to $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$ discussed earlier, there are other bilinear covariants. One can show that the following 16 bilinear covariants are independent and complete:

$$\begin{array}{lll} \bar{\psi}\psi & (1) & \text{scalar} \\ \bar{\psi}\gamma^\mu\psi & (4) & \text{vector} \\ \bar{\psi}\sigma^{\mu\nu}\psi & (6) & \text{tensor} \\ \bar{\psi}\gamma^5\psi & (1) & \text{pseudoscalar} \\ \bar{\psi}\gamma^5\gamma^\mu\psi & (4) & \text{axial vector} \end{array} \tag{4.136}$$

where $$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \gamma^5 = \gamma^{5^+} \tag{4.137}$$
γ^5 anticommutes with all γ^μ

$$\left[\gamma^5,\ \gamma^\mu\right] = 0 \tag{4.138}$$

γ^5 commutes with proper Lorentz transformation S

$$\gamma^5 S = S\gamma^5$$

but anticommutes with space inversion S_P

$$\gamma^5 S_P = S_P \gamma^5 \tag{4.140}$$

The transformation of the 16 bilinear covariants under proper Lorentz transformations and space inversion, as given by Equation 4.136, can be readily proved using Equations 4.53, 4.74, 4.139 and 4.140.

The γ^5 matrix is often encountered in the context of chirality. Although $\left[H,\ \gamma^5\right] \neq 0$, it can be shown that at high energies E >> m

$$\gamma^5 u = \begin{bmatrix} \vec{\sigma}\bullet\vec{P} & 0 \\ 0 & \vec{\sigma}\bullet\vec{P} \end{bmatrix} u$$

and $\frac{1-\gamma^5}{2}, \frac{1+\gamma^5}{2}$ projects the left and right-handed components of the Dirac spinor u.

We have based our discussion on the Dirac equation upon the Pauli-Dirac representation of the γ matrices. How would the wave function be modified if we use different representations for the γ matrices?

Suppose we use another representation $\gamma^{\mu\prime}$ for the Dirac equation with wave function ψ':

$$\left(i\gamma^{\mu\prime}\partial_\mu - m\right)\psi' = 0 \quad \text{where} \quad \left[\gamma^{\mu\prime},\ \gamma^{\nu\prime}\right] = 2g^{\mu\nu} \tag{4.141}$$

How does ψ' differ from ψ in the 'regular' Dirac equation?

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0 \tag{4.142}$$

The 'Pauli's fundamental theorem' shows that for any $\gamma^{\mu\prime}$ satisfying the relations $\left[\gamma^{\mu\prime},\ \gamma^{\nu\prime}\right] = 2g^{\mu\nu}$, a transformation S can be found such that

$$S\gamma^{\mu}S^{-1} = \gamma^{\mu\prime} \tag{4.143}$$

Equation 4.141 becomes

$$\left(i\ S\gamma^{\mu}S^{-1}\partial_{\mu} - m\right)S\ S^{-1}\psi' = 0 \tag{4.144}$$

multiplying S^{-1} from the left gives

$$\left(i\ \gamma^{\mu}\partial_{\mu} - m\right)S^{-1}\ \psi' = 0 \tag{4.145}$$

Equation 4.145 is equivalent to Equation 4.142 provided

$$\psi = S^{-1}\psi' \quad \text{or} \quad \psi' = S\psi \tag{4.146}$$

Therefore, the wave functions obtained using different representations of the γ-matrices are related by the same transformation S used to relate γ^{μ} and $\gamma^{\mu\prime}$.

We complete our discussion on the Dirac equation by presenting an alternative way to derive the Dirac equation.

We can start with the relativistic energy-momentum relation

$$\mathrm{E}^2 - P^2 = m^2 \tag{4.147}$$

or equivalently

$$\mathrm{E}^2 - \left(\vec{\sigma}\bullet\vec{P}\right)^2 = m^2 \tag{4.148}$$

Making the substitution $\mathrm{E} \to i\frac{\partial}{\partial t}, \quad \vec{P} \to -i\vec{\nabla}$ and acting on wave function ϕ, we have

$$\left(i\frac{\partial}{\partial t} + i\vec{\sigma}\bullet\vec{\nabla} \right)\left(i\frac{\partial}{\partial t} - i\vec{\sigma}\bullet\vec{\nabla} \right)\phi = m^2\phi \qquad (4.149)$$

Now ϕ is a two-component wave function.
Define

$$\begin{aligned} \phi^{(R)} &= \frac{1}{m}\left(i\frac{\partial}{\partial t} - i\vec{\sigma}\bullet\vec{\nabla} \right)\phi \\ \phi^{(L)} &= \phi \end{aligned} \qquad (4.150)$$

then Equation 4.149 gives

$$\begin{aligned} \phi^{(R)} &= \frac{1}{m}\left(i\frac{\partial}{\partial t} - i\vec{\sigma}\bullet\vec{\nabla} \right)\phi^{(L)} \\ \phi^{(L)} &= \frac{1}{m}\left(i\frac{\partial}{\partial t} + i\vec{\sigma}\bullet\vec{\nabla} \right)\phi^{(R)} \end{aligned} \qquad (4.151)$$

We can now construct a 4-component wave function ψ

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} \phi^{(R)} + \phi^{(L)} \\ \phi^{(R)} - \phi^{(L)} \end{pmatrix} \qquad (4.152)$$

From Equation 4.151, one can form the following equations:

$$\phi^{(R)} + \phi^{(L)} = \frac{1}{m}\left[i\frac{\partial}{\partial t}\left(\phi^{(R)} + \phi^{(L)}\right) + i\vec{\sigma}\bullet\vec{\nabla}\left(\phi^{(R)} - \phi^{(L)}\right) \right] \qquad (4.153)$$

$$\phi^{(R)} - \phi^{(L)} = \frac{1}{m}\left[-i\frac{\partial}{\partial t}\left(\phi^{(R)} - \phi^{(L)}\right) + i\vec{\sigma}\bullet\vec{\nabla}\left(\phi^{(R)} + \phi^{(L)}\right) \right] \qquad (4.154)$$

From Equation 4.152, Equations 4.153, 4.154 can be expressed

$$\begin{pmatrix} i\frac{\partial}{\partial t} & i\vec{\sigma}\bullet\vec{\nabla} \\ -i\vec{\sigma}\bullet\vec{\nabla} & -i\frac{\partial}{\partial t} \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = m\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \tag{4.155}$$

which is simply

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0 \tag{4.156}$$

the Dirac equation.