Chapter 3

Building Hadrons from Quarks

Mesons in SU(2)

We are now ready to consider mesons and baryons constructed from quarks. As we recall, mesons are made of quark-antiquark pair and baryons are made of three quarks.

Consider mesons made of $u$ and $d$ quarks first.

$$\psi = \psi (\text{space}) \psi (\text{spin}) \psi (\text{flavor}) \psi (\text{color})$$

It is straightforward to write down the spin wave function of a meson. Since both $q$ and $\bar{q}$ have spin $\frac{1}{2}$, the $qq\bar{q}$ can form either a spin-triplet or a spin-singlet state:

$$|1,1\rangle = \uparrow \uparrow$$
$$|1,0\rangle = \frac{1}{\sqrt{2}} (\uparrow \downarrow + \downarrow \uparrow)$$ spin 1 triplet
$$|1,-1\rangle = \downarrow \downarrow$$

$$|0,0\rangle = \frac{1}{\sqrt{2}} (\uparrow \downarrow - \downarrow \uparrow)$$ spin 0 singlet

For the flavor wave function, we have an analogous situation. The $u, d$ quarks are isospin doublet, and so are $d, u$. However, a slight complication arises since $\bar{d}, \bar{u}$ transforms in the complex conjugate representation.

Now

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = I_i \begin{pmatrix} u \\ d \end{pmatrix}$$

and

$$\left[ I_i, I_j \right] = i \varepsilon_{ijk} I_k$$

taking complex conjugate, we get

$$\left[ I_i^*, I_j^* \right] = -i \varepsilon_{ijk} I_k^*$$

or

$$\left[ -I_i, -I_j \right] = i \varepsilon_{ijk} (-I_k^*)$$
\(-I_i^*\) transforms \(\bar{u}, \bar{d}\) into \(\bar{u}', \bar{d}'\):

\[
\begin{pmatrix}
\bar{u}' \\
\bar{d}'
\end{pmatrix} = (-I_i^*) \begin{pmatrix}
\bar{u} \\
\bar{d}
\end{pmatrix} \tag{Eq. 1}
\]

write out \(-I_i^*\) explicitly:

\[
(-I_1^*) = \frac{1}{2} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \quad (-I_2^*) = \frac{1}{2} \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} \quad (-I_3^*) = \frac{1}{2} \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\]

One can easily verify that \(-I_i^*\) is related to \(I_i\) by a unitary transformation

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} (-I_i^*) \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} = I_i
\]

One can multiply (Eq. 1) by \(\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}\) on the left

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
\bar{u}' \\
\bar{d}'
\end{pmatrix} = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} (-I_i^*) \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
\bar{u} \\
\bar{d}
\end{pmatrix} \tag{Eq. 2}
\]

(Eq. 2) becomes

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
\bar{u}' \\
\bar{d}'
\end{pmatrix} = I_i \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
\bar{u} \\
\bar{d}
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
-\bar{d}' \\
\bar{u}'
\end{pmatrix} = I_i \begin{pmatrix}
-\bar{d} \\
\bar{u}
\end{pmatrix}
\]

Therefore, \(\begin{pmatrix}
-\bar{d} \\
\bar{u}
\end{pmatrix}\) transforms just like \(\begin{pmatrix}
u \\
d
\end{pmatrix}\) and is the isospin doublet for \(\bar{u}, \bar{d}\).
Combining the \( \begin{pmatrix} u \\ d \end{pmatrix} \) with the \( \begin{pmatrix} -d \\ u \end{pmatrix} \), one obtains isospin triplet and isospin singlet:

\[
|I, I_z\rangle = |1, 1\rangle = -u\bar{d} \\
|1, 0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\
|1, -1\rangle = \bar{u}d \\
|0, 0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})
\]

The total spin of the mesons can be \( S = 0 \) or \( 1 \). The lightest mesons have orbital angular momentum \( L = 0 \). Therefore, their total angular momentum \( J = 0 \) or \( 1 \).

- \( J = 0, I = 1 \) : \( \pi^+, \pi^o, \pi^- \) (mass \( \sim 140 \text{ MeV} \))
- \( J = 1, I = 1 \) : \( \rho^+, \rho^o, \rho^- \) (mass \( \sim 770 \text{ MeV} \))
- \( J = 1, I = 0 \) : \( \omega^o \) (mass \( \sim 780 \text{ MeV} \))

(We will postpone the discussion of \( J = 0, I = 0 \) meson until later, since its wave function contains non-\( u, d \) components.)

One can readily combine the flavor and spin parts of the wavefunction. For example:

\[
\rho^o (J_z = 0) = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) = \frac{1}{2}(u \uparrow \bar{u} \downarrow + u \downarrow \bar{u} \uparrow - d \uparrow \bar{d} \downarrow - d \downarrow \bar{d} \uparrow)
\]

\[
\pi^+ (J_z = 0) = -u\bar{d} \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) = -\frac{1}{\sqrt{2}}(u \uparrow \bar{d} \downarrow - u \downarrow \bar{d} \uparrow)
\]

**Baryons in SU(2)**

Now, consider baryons made of \( u, d \) quarks. Let us start with the flavor part.
First, combining two quarks:

\[
\begin{align*}
|1,1\rangle & : uu \\
|1,0\rangle & : \frac{1}{\sqrt{2}}(ud + du) \\
|1,-1\rangle & : dd \\
|0,0\rangle & : \frac{1}{\sqrt{2}}(ud - du)
\end{align*}
\]

\(I = 1\) triplet

\(I = 0\) singlet

Note that \(I = 1\) is symmetric with respect to the interchange of the two quarks, while \(I = 0\) singlet is anti-symmetric. We therefore label the symmetry property of the representations as

\[2 \otimes 2 = 3_S \oplus 1_A\]

where the dimension of the representation is \(2I+1\) (For example, \(I=1/2 \rightarrow 2; I=1 \rightarrow 3; I=0 \rightarrow 1\)). Adding another \(u/d\) quark, we have

\[2 \otimes 2 \otimes 2 = (3_S + 1_A) \otimes 2 = (3_S \otimes 2) \oplus (1_A \otimes 2)\]

First consider \(1_A \otimes 2\):

\[
\frac{1}{\sqrt{2}}(ud - du) \otimes \begin{pmatrix} u \\ d \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}}(ud - du)u \\ \frac{1}{\sqrt{2}}(ud - du)d \end{pmatrix}
\]

This is an isospin doublet with mixed permutation symmetry (anti-symmetric WRT the interchange of the first two quarks). Therefore, we denote

\[1_A \otimes 2 = 2_{M_A}\]

where \(M\) signifies ‘mixed-symmetry’ while the subscript \(A\) reminds us the anti-symmetry between the first two quarks.
We now consider the remaining part, $3_s \otimes 2$:

\[
\begin{pmatrix}
    uu \\
    \frac{1}{\sqrt{2}} (ud + du) \\
    dd
\end{pmatrix}
\otimes
\begin{pmatrix}
    u \\
    d
\end{pmatrix}
\quad (I = 1 \otimes I = \frac{1}{2} \Rightarrow I = \frac{3}{2} \oplus I = \frac{1}{2})
\]

This simply corresponds to the product of an isospin triplet and an isospin doublet. The resulting isospin can have $I = \frac{3}{2}$ and $I = \frac{1}{2}$.

$I = \frac{3}{2}$:

\[
\begin{pmatrix}
    uuu \\
    \frac{1}{\sqrt{3}} uud + \sqrt{\frac{2}{3}} \frac{1}{\sqrt{2}} (ud + du) u \\
    \sqrt{\frac{2}{3}} \frac{1}{\sqrt{2}} (ud + du) d + \sqrt{\frac{1}{3}} ddu \\
    ddd
\end{pmatrix}
\begin{pmatrix}
    uuu \\
    uud + udu + duu \\
    udd + du + ddu \\
    ddd
\end{pmatrix}
\]

$I = \frac{1}{2}$:

\[
\begin{pmatrix}
    \frac{\sqrt{2}}{3} uud - \sqrt{\frac{1}{3}} \frac{1}{\sqrt{2}} (ud + du) u \\
    \sqrt{\frac{2}{3}} \frac{1}{\sqrt{2}} (ud + du) d - \sqrt{\frac{2}{3}} ddu \\
\end{pmatrix}
\begin{pmatrix}
    \frac{1}{\sqrt{6}} (2uud - udu - duu) \\
    \frac{1}{\sqrt{6}} (udd + dud - 2ddu)
\end{pmatrix}
\]

In constructing the $I = \frac{3}{2}$ and $I = \frac{1}{2}$ multiplets, we simply use the Clebsch-Gordon coefficients to combine the $|I_1, I_{1z}, I_2, I_{2z}\rangle$ states into $|I, I_z\rangle$ states.
$I = \frac{3}{2}$ multiplet is totally symmetric WRT interchange of any pair, while $I = \frac{1}{2}$ multiplet has mixed-symmetry and is symmetric WRT the interchange of the first two quarks.

Therefore, we write

$$3_s \otimes 2 = 4_s \oplus 2_{M_s}$$

Collecting previous result, we now have

$$2 \otimes 2 \otimes 2 = 4_s \oplus 2_{M_s} \oplus 2_{M_A}$$

The spin wave functions for three quarks are constructed exactly the same way.

The flavor-spin wave functions of the $\Delta^{++}$, $\Delta^+$, $\Delta^0$, $\Delta^-$ quartet are constructed by combining the $I = \frac{3}{2}$ (flavor) with the $J = \frac{3}{2}$ (spin) wave function.

$$\Delta: (4_s, 4_s)$$

For example $\Delta^{++} \left(J_z = \frac{3}{2}\right)$ is simply $(uuu)(\uparrow\uparrow\uparrow) = u \uparrow u \uparrow u \uparrow$

while $\Delta^+ \left(J_z = \frac{1}{2}\right)$ is

$$\frac{1}{\sqrt{3}}(uud + udu + duu) \frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)
= \frac{1}{3}[(u \uparrow u \uparrow d \downarrow + u \uparrow u \downarrow d \uparrow + u \downarrow u \uparrow d \downarrow) + \text{permutations}]$$

Note that the $\Delta$ flavor-spin wave function is totally symmetric with respect to exchange of any pair of quarks. Since quarks are fermions, the wave function should be anti-symmetric overall. We now believe the color wave function is anti-symmetric (in order to become a color-singlet state), making the full wave function anti-symmetric.

The proton and neutron, being spin-$\frac{1}{2}$ isospin-$\frac{1}{2}$ particles, are constructed with the following combination to make the overall flavor-spin wave function symmetric.
As an example, consider proton with \( J_Z = \frac{1}{2} \):

\[
p(J_Z = \frac{1}{2}) = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{6}} (2uud - udu - duu) \cdot \frac{1}{\sqrt{6}} (2 \uparrow \downarrow \downarrow - \downarrow \uparrow \uparrow - \downarrow \uparrow \uparrow) + \frac{1}{\sqrt{2}} (udu - duu) \frac{1}{\sqrt{2}} (\uparrow \uparrow \uparrow - \downarrow \uparrow \uparrow) \right]
\]

\[
= \frac{1}{\sqrt{18}} \left[ uud (2 \uparrow \downarrow \downarrow - \downarrow \uparrow \uparrow - \downarrow \uparrow \uparrow) + udu (2 \uparrow \downarrow \downarrow - \uparrow \uparrow \downarrow - \downarrow \uparrow \uparrow) + duu (\downarrow \uparrow \uparrow - \uparrow \downarrow \uparrow - \uparrow \uparrow \downarrow) \right]
\]

\[
= \frac{1}{\sqrt{18}} \left\{ \left[ 2u \uparrow u \uparrow d \downarrow - u \uparrow u \downarrow d \uparrow - u \downarrow u \uparrow d \uparrow \right] + \text{permutations} \right\}
\]

Note that \( p(J_Z = -\frac{1}{2}) \) can be obtained from \( p(J_Z = +\frac{1}{2}) \) by interchanging \( \uparrow \) with \( \downarrow \).

Similarly, \( n(J_Z = \frac{1}{2}) \) can be obtained from \( p(J_Z = \frac{1}{2}) \) by interchanging \( u \) with \( d \) quarks.

**SU(3)**

Adding strange quark, we extend SU(2) to SU(3):

\[
\begin{pmatrix} u \\ d \end{pmatrix} \Rightarrow \begin{pmatrix} u \\ d \\ s \end{pmatrix}
\]

SU(\( N \)) has \( N^2 - 1 \) generators, and the fundamental representation has matrices of dimension \( N \). The eight generators for SU(3) satisfy the Lie algebra:
where the structure constants are

\[ f_{i12} = 1, f_{i47} = f_{i246} = f_{i257} = f_{i345} = f_{i516} = f_{i637} = \frac{1}{2}, f_{i458} = f_{i678} = \sqrt{3}/2 \]

The structure constants are anti-symmetric with respect to interchange of any two indices.

The fundamental representations for SU(3) are:

\[
\begin{align*}
T_1 &= \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
T_2 &= \frac{1}{2} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
T_3 &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
T_4 &= \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
T_5 &= \frac{1}{2} \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \\
T_6 &= \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
T_7 &= \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \\
T_8 &= \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}
\end{align*}
\]

Mesons in SU(3)

\(u, d, s\) quarks belong to the fundamental representation of SU(3), denoted by 3. \(\bar{u}, \bar{d}, \bar{s}\) belong to the conjugate representation \(\bar{3}\). Combining quark and anti-quark together:
\[ 3 \otimes \bar{3} = 8 \oplus 1 \]

In terms of the ‘weight’ diagram, the 3 and \( \bar{3} \) representations are:

Where \( Y = B + S \), \( B \) is the baryon number and \( S \) is the strangeness.

\( 3 \otimes \bar{3} \) is obtained by superimposing the \( \bar{3} \) triplet on top of each site of the quark 3 triplet. One obtains

The SU(3) singlet must contain \( s, u, d \) quarks equally and has the wave function

\[ \sqrt{\frac{1}{3}} (u\bar{u} + d\bar{d} + s\bar{s}) \]

The other two mesons occupying the \( I_3 = 0, Y = 0 \) location are the SU(2) triplet state.
and the SU(2) singlet state

\[
\frac{1}{\sqrt{6}} (u\bar{u} + d\bar{d} - 2s\bar{s})
\]

The lightest meson octet consists of \( K^+ \), \( K^0 \), \( \pi^+ \), \( \pi^0 \), \( \eta \), \( \eta' \) and \( \epsilon \). Together with the SU(3) singlet \( \eta' \), they form the pseudoscalar \((0^-)\) nonet.

The \( J = 1 \) meson octet consists of:

and the \( J = 1 \) singlet is \( \phi \).

The \( \omega \) and \( \phi \) mesons are found to be mixtures of the SU(3) states \( \omega_8, \phi_1 \)

\[
\begin{align*}
\phi &= -\frac{\sqrt{2}}{3} \omega_8 + \frac{1}{3} \phi_1 = s\bar{s} \\
\omega &= \frac{1}{\sqrt{3}} \omega_8 + \frac{2}{3} \phi_1 = \frac{1}{\sqrt{2}} (u\bar{u} + d\bar{d})
\end{align*}
\]

Evidence for \( \phi \) to be nearly a pure \( s\bar{s} \) state is from the peculiar behavior of \( \phi \) decays.
\( \phi \) has a width of only 4.3 MeV and its relatively long lifetime is due to the fact that the main decay channels for \( \phi \) are

\[
\phi \rightarrow K^+ K^-, \ K_L^0 K_S^0 \quad \text{(Branching Ratio } \approx 83\%)\]

The phase space for \( \phi \) (1020) decaying into a pair of kaons is greatly suppressed relative to the

\[
\phi \rightarrow \pi^+ \pi^- \pi^0 \quad \text{(Branching Ratio } \approx 16\%)\]

However, the \( \phi \rightarrow 3\pi \) decay is suppressed by the Zweig rule if \( \phi \) is mainly a \( s\bar{s} \) state. The \( \phi \rightarrow KK \) decays, which are not affected by the Zweig rule, became the dominant decay channels.

The Zweig rule (or OZI rule, due to Okubo, Zweig, and Iizuka) states that decay rates for processes described by diagrams with unconnected quark lines are suppressed. Thus, \( \phi \rightarrow \pi^+ \pi^- \pi^0 \) decay diagram

shows that the 3\( \pi \) quark lines are completely disconnected from the initial \( s\bar{s} \) quark lines. In contrast, the \( \phi \rightarrow K^+ K^- \) decay diagram has some quark lines connected, and it is not suppressed by the Zweig rule.
The three colors ($R$, $B$, $G$) of quark transform under the SU(3) exact symmetry. Since mesons are made of $q - \bar{q}$, their color wave function should be an SU(3)$_c$ singlet. The singlet state in $3_c \otimes \overline{3}_c = 8_c \oplus 1_c$ is simply

$$\frac{1}{\sqrt{3}} (R\bar{R} + B\bar{B} + G\bar{G})$$

Therefore, the color wave functions of all mesons have the same form as the flavor wave function of an SU(3) singlet meson ($\eta'$).

As discussed earlier, the color contents of gluons are (color x color). Recall the quark-gluon coupling diagram:

```
q_R \quad G_{\bar{B}} \quad q_B
```

for a ‘$R$’ quark changing into a ‘$B$’ quark and the gluon carrying $R\bar{B}$ color. The gluons therefore have eight different color combinations:

$$R\bar{G}, \; R\bar{B}, \; G\bar{R}, \; G\bar{B}, \; B\bar{R}, \; B\bar{G}, \; \sqrt{\frac{1}{2}} (R\bar{R} - G\bar{G}), \; \sqrt{\frac{1}{6}} (R\bar{R} + G\bar{G} - 2B\bar{B})$$

Note that the color contents of the 8 gluons are analogous to the flavor contents of the meson octet.

**Baryons in SU(3)**

The spin wave functions for baryons have been discussed earlier:

$$2 \otimes 2 \otimes 2 = (3 \oplus 1) \otimes 2 = 4_s \oplus 2_{M_s} \oplus 2_{M_A}$$

For the flavor wave function, we first consider $3 \otimes 3$
\( \begin{pmatrix} u \\ d \\ s \end{pmatrix} \otimes \begin{pmatrix} u \\ d \\ s \end{pmatrix} \) gives 9 basis states: \( uu, ud, us, du, dd, ds, su, sd, ss \)

These basis states can be rearranged to form basis states with specific permutation symmetry. In particular, the basis states \( \psi_i \varphi_j \) \( (i = u, d, s; j = u, d, s) \) can be expressed as

\[
\begin{aligned}
\psi_i \varphi_j &= \frac{1}{2} \left( \psi_i \varphi_j + \psi_j \varphi_i \right) + \frac{1}{2} \left( \psi_i \varphi_j - \psi_j \varphi_i \right) \\
&= \text{symmetric} \quad \text{anti-symmetric}
\end{aligned}
\]

It is clear that the symmetric \( S_{ij} \) contains 6 elements, while the anti-symmetric \( A_{ij} \) contains 3 elements

\[
3 \otimes 3 = 6_s \oplus 3_A
\]

In terms of the SU(3) weight diagram
Now $3 \otimes 3 \otimes 3 = (6_s \oplus \overline{3}_A) \otimes 3 = (6_s \otimes 3) \oplus (\overline{3}_A \otimes 3)$

First consider $6_s \otimes 3$

\[
S_{ij \psi k} = \frac{1}{2} \left( S_{ij \psi k} + S_{kj \psi i} + S_{ki \psi j} \right) + \frac{1}{2} \left( S_{ij \psi k} - S_{kj \psi i} - S_{ki \psi j} \right)
\]

The first part of the decomposition is totally symmetric and contains 10 elements, while the second part contains 8 elements and has mixed symmetry (symmetric only for $i \leftrightarrow j$ interchange).

Therefore $6_s \otimes 3 = 10_s \oplus 8_{M_S}$

Next consider $\overline{3}_A \otimes 3$

\[
A_{ij \psi k} = \frac{1}{2} \left( A_{ij \psi k} + A_{jk \psi i} + A_{ki \psi j} \right) + \frac{1}{2} \left( A_{ij \psi k} - A_{jk \psi i} - A_{ki \psi j} \right)
\]

The first term has 1 element and is totally anti-symmetric, while the second term has 8 elements with mixed symmetry (anti-symmetric only for $i \leftrightarrow j$ interchange).

Therefore $\overline{3}_A \otimes 3 = 8_{M_A} \oplus 1_A$

Finally we have $3 \otimes 3 \otimes 3 = 10_s \oplus 8_{M_S} \oplus 8_{M_A} \oplus 1_A$

The explicit flavor wave functions for $10_s$, $8_{M_S}$, $8_{M_A}$, and $1_A$ are listed on the next two pages.
SU(3) Baryons

Flavor: \[ 3 \otimes 3 = 10_s \oplus 8_{M_s} \oplus 8_{M_A} \oplus 1_A \]

10_s:
\[ \Delta^{++}: uuu \]
\[ \Delta^+: \frac{1}{\sqrt{3}}(uud + udu + duu) \]
\[ \Delta^0: \frac{1}{\sqrt{3}}(ddu + dud + udd) \]
\[ \Delta^-: ddd \]
\[ \Sigma^+: \frac{1}{\sqrt{3}}(uus + usu + suu) \]
\[ \Sigma^0: \frac{1}{\sqrt{6}}(uds + usd + dsu + dus + sud + sdu) \]
\[ \Sigma^-: \frac{1}{\sqrt{3}}(dds + dsd + sdd) \]
\[ \Xi^0: \frac{1}{\sqrt{3}}(ssu + sus + uss) \]
\[ \Xi^-: \frac{1}{\sqrt{3}}(ssd + sds + dss) \]
\[ \Omega^-: sss \]

8_{M_s}:
\[ p: \frac{1}{\sqrt{6}}(2uud - udu - duu) \]
\[ n: \frac{1}{\sqrt{6}}(-2ddu + dud + udd) \]
\[ \Sigma^+: \frac{1}{\sqrt{6}}(2uus - usu - suu) \]
\[ \Sigma^0: \frac{1}{\sqrt{12}}(2uds - usd - dsu + 2dus - sud - sdu) \]
\[ \Sigma^-: \frac{1}{\sqrt{6}}(2dds - dsd - sdd) \]
\[ \Lambda^0: \frac{1}{2}(usd + sud - sdu - dsu) \]
\[ \Xi^0: \frac{1}{\sqrt{6}}(sus + uss - 2ssu) \]
\[ \Xi^-: \frac{1}{\sqrt{6}}(sds + dss - 2ssd) \]
\[ 8_{M_A} : \]
\[ p : \frac{1}{\sqrt{2}} (udu - duu) \]
\[ n : \frac{1}{\sqrt{2}} (udd + dud) \]
\[ \Sigma^+ : \frac{1}{\sqrt{2}} (usu - suu) \]
\[ \Sigma^- : \frac{1}{\sqrt{2}} (dsd - sdd) \]
\[ \Lambda^o : \frac{1}{\sqrt{12}} (2uds - dsu - sud - 2dus - sdu + usd) \]
\[ \Xi^o : \frac{1}{\sqrt{2}} (uss - sus) \]
\[ \Xi^- : \frac{1}{\sqrt{2}} (dss - sds) \]
\[ 1_A : \frac{1}{\sqrt{6}} (uds - usd + dsu - dus + sud - sdu) \]

Note \( 8_{M_S} \) is obtained by the following sequence:

starting from \( q_1 q_2 q_3 \)

a) symmetrize in positions 1 and 2
b) anti-symmetrize in positions 1 and 3
c) symmetrize in positions 1 and 2

For \( \Lambda^o \), orthogonality to \( \Sigma^o, \Sigma^o^* \), and \( 1_A \) are imposed to obtain its wave function.

For \( 8_{M_A} \), the procedure is analogous, but with anti-symmetrization in steps a) and c) and symmetrization in step b). Also, \( \Sigma^o \) is obtained by requiring orthogonality to \( \Lambda^o, \Sigma^o^* \) and \( 1_A \).
Spin:

\[ 2 \otimes 2 \otimes 2 = 4_s \oplus 2_{M_s} \oplus 2_{M_d} \]

4\(_s\):

\[ |\frac{3}{2}, \frac{3}{2}\rangle: \quad \uparrow\uparrow\uparrow \]

\[ |\frac{3}{2}, \frac{1}{2}\rangle: \quad \frac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow + \uparrow\uparrow\downarrow + \downarrow\uparrow\uparrow) \]

\[ |\frac{3}{2}, -\frac{1}{2}\rangle: \quad \frac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow) \]

\[ |\frac{3}{2}, -\frac{3}{2}\rangle: \quad \downarrow\downarrow \]

2\(_{M_s}\):

\[ |\frac{1}{2}, \frac{1}{2}\rangle: \quad \frac{1}{\sqrt{6}}(2 \uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \]

\[ |\frac{1}{2}, -\frac{1}{2}\rangle: \quad \frac{1}{\sqrt{6}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow - \downarrow\downarrow\uparrow) \]

2\(_{M_d}\):

\[ |\frac{1}{2}, \frac{1}{2}\rangle: \quad \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\downarrow) \]

\[ |\frac{1}{2}, -\frac{1}{2}\rangle: \quad \frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow) \]

Baryon decuplet: \((10\(_S\), 4\(_S\))\)

Baryon octet:

\[ \frac{1}{\sqrt{2}}[(8\(_{M_s}\), 2\(_{M_s}\)) + (8\(_{M_d}\), 2\(_{M_d}\))] \]

We can write down explicitly the flavor-spin wave functions of the baryon octet:

\[ p\uparrow = \frac{1}{\sqrt{18}}\left\{[2u \uparrow u \uparrow d \downarrow -u \uparrow u \downarrow d \uparrow -u \downarrow u \uparrow d \downarrow] + \text{permutations}\right\} \]

\[ n\uparrow = \frac{1}{\sqrt{18}}\left\{[-2d \uparrow d \uparrow u \downarrow +d \uparrow d \downarrow u \uparrow +d \downarrow d \uparrow u \uparrow] + \text{permutations}\right\} \]

\[ \Sigma^+\uparrow = \frac{1}{\sqrt{18}}\left\{[2u \uparrow u \uparrow s \downarrow -u \uparrow u \downarrow s \uparrow -u \downarrow u \uparrow s \uparrow] + \text{permutations}\right\} \]

\[ \Sigma^0\uparrow = \frac{1}{6}\left\{[2u \uparrow u \uparrow d \downarrow -u \uparrow d \downarrow s \uparrow -u \downarrow d \uparrow s \uparrow] + \text{permutations}\right\} \]

\[ \Sigma^-\uparrow = \frac{1}{\sqrt{18}}\left\{[2d \uparrow d \uparrow s \downarrow -d \uparrow d \downarrow s \uparrow -d \downarrow d \uparrow s \uparrow] + \text{permutations}\right\} \]

\[ \Xi^-\uparrow = \frac{1}{\sqrt{18}}\left\{[-2s \uparrow s \uparrow u \downarrow +s \uparrow s \downarrow u \uparrow +s \downarrow s \uparrow u \uparrow] + \text{permutations}\right\} \]
Note the following for the flavor-spin wave functions of the baryon octet:

1) For $p$, $n$, $\Sigma^+$, $\Sigma^-$, $\Xi^0$, $\Xi^-$, they contain two identical quarks and there are three permutations, i.e.

$$
\Xi^- \uparrow = \frac{1}{\sqrt{18}} \left\{ \left[ -2s \uparrow s \uparrow d \downarrow + s \uparrow s \downarrow d \uparrow + s \downarrow s \uparrow d \uparrow \right] + \text{permutations} \right\}
$$

$$
\Lambda^o \uparrow = \frac{1}{\sqrt{12}} \left\{ \left[ u \uparrow d \downarrow s \uparrow - u \downarrow d \uparrow s \uparrow \right] + \text{permutations} \right\}
$$

For $\Sigma^o$ and $\Lambda$, they have three different quarks, and there are six permutations, i.e.

$$
\Lambda^o \uparrow = \frac{1}{\sqrt{12}} \left\{ \left[ u \uparrow d \downarrow s \uparrow - u \downarrow d \uparrow s \uparrow \right] + \left[ d \downarrow u \uparrow s \uparrow - d \uparrow u \downarrow s \uparrow \right] + \left[ s \uparrow d \downarrow u \uparrow - s \downarrow d \uparrow u \uparrow \right] + \left[ s \uparrow u \uparrow d \downarrow - s \uparrow u \downarrow d \uparrow \right] + \left[ s \uparrow u \uparrow d \downarrow - s \uparrow u \downarrow d \uparrow \right] + \left[ s \uparrow u \uparrow d \downarrow - s \uparrow u \downarrow d \uparrow \right] \right\}
$$

2) Once one knows the proton wave function, one can obtain the wave functions for other baryons by replacing (interchanging) the quark flavors. The resulting wave function is correct within an overall factor.

$$
p \leftrightarrow n \ (u \leftrightarrow d) \quad p \leftrightarrow \Sigma^+ \ (d \leftrightarrow s) \quad \Sigma^+ \leftrightarrow \Sigma \ (u \leftrightarrow d) \quad n \leftrightarrow \Xi^0 \ (d \leftrightarrow s) \quad \Xi^o \leftrightarrow \Xi^- \ (u \leftrightarrow d)
$$

3) The total spin of an identical quark pair is always in an $s = 1$ (symmetric) state. This is required since the overall wave function is anti-symmetric for this quark pair, and the color wave function is anti-symmetric.

4) For $\Sigma^o$, the $ud$ have $s = 1$, while $s = 0$ for $ud$ in $\Lambda^o$. Also, for $\Lambda^o$ the baryon’s spin only resides on the strange quark.