Chapter 6

Nucleon Structure and the Parton Model

1. Electron-Proton Elastic Scattering

Electron scattering provides the most powerful tool for revealing the internal structure of the nucleon. Much of the theoretical background for understanding the formulations in electron scattering has already been discussed in the last chapter. We begin by summarizing the relevant cross sections corresponding to different assumptions used in treating the \( e^-p \) scattering.

1) Spin-0 Particle Scattering off a Static Point Spin-0 Particle with Charge \( e \)

Recall Equation 5.45

\[
\left( \frac{d\sigma}{d\Omega} \right)_{cm} = \frac{\alpha^2}{4s} \frac{P_f}{P_i} \left( s - t \right)^2
\]  

(6.1)

for scattering of spin-0 particle off a spin-0 particle. A static target is represented by \( M \gg m \), where \( m, M \) are the incident and target mass respectively.

\[
P_a = (E_a, \vec{K})
\]

\[
P_b = (M, 0) = P'_b
\]

\[
P'_a = (E'_a, \vec{K}')
\]

\[
s = (P_a + P_b)^2 = P_a^2 + 2E_aM + M^2
\]

\[
u = (P_a - P'_b)^2 = P_a^2 - 2E'_aM + M^2
\]

For \( M \gg m \), \( |K| = |K'| \), \( E_a = |K| \), \( E'_a = |K'| \), \( E_a = E'_a \).
Hence Equation 6.1 becomes

\[
\left( \frac{d\sigma}{d\Omega} \right)_{cm} = \frac{\alpha^2}{4M^2} \left( \frac{4|K|M}{\sin^2\frac{\theta}{2}} \right)^2 = \frac{\alpha^2}{4|K|^2 \sin^4\frac{\theta}{2}}
\]  \quad (6.2)

which is recognized as the Rutherford scattering.

2) Spin-½ Electron Scattering off a Static Spin-0 Point-Charge \( e \)

We have already derived the cross section for this case. The expression is given in Equation 5.79. Namely,

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2 E^2}{4|K|^4 \sin^4\frac{\theta}{2}} \left( 1 - v^2 \sin^2 \frac{\theta}{2} \right)
\]  \quad (6.3)

This is the Mott scattering. Note that at high energy \( v \to 1 \), and the Mott scattering becomes

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2 E^2}{4|K|^4 \sin^4\frac{\theta}{2}} \cos^2 \frac{\theta}{2}
\]  \quad (6.4)

and scattering to \( 180^\circ \) is forbidden. This can be understood by noting that helicity of the electron is conserved at high energy. The following illustration

<table>
<thead>
<tr>
<th>Initial</th>
<th>Final</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^- )</td>
<td>( e^- )</td>
</tr>
</tbody>
</table>
shows that for electron scattered to $180^\circ$, the spin has to be flipped to conserve helicity. Since a spin-0 target cannot flip the electron spin, such scattering is forbidden.

3) Spin-$\frac{1}{2}$ Electron Scattering off a Spin-0 Static Composite Particle

The cross section in this case is worked out in Halzen and Martin, Ex. 8.1. The result is

$$\left( \frac{d\sigma}{d\Omega} \right) = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} |F(\vec{q})|^2$$

where $F(\vec{q})$ is the Fourier transform of the charge distribution $\rho(x)$:

$$F(\vec{q}) = \int d^3x \rho(x) e^{i\vec{q}\cdot\vec{x}}$$

4) Spin-$\frac{1}{2}$ Electron Scattering off a Spin-0 Point-like Particle which can Recoil

This case can be worked out by replacing the $L_{\mu\nu}^{\text{munu}}$ which spears in the $e^+\mu^- \rightarrow e^+\mu^-$ scattering by the $(p + p')_\mu (p + p')_\nu$ corresponding to the spin-0 vertex. The result, as shown in Ex. 6.8 of Halzen and Martin, is

$$\left( \frac{d\sigma}{d\Omega} \right) = \left( \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \right) \frac{E'}{E} \cos^2 \frac{\theta}{2}$$

where one assumes the electron mass is negligible. A comparison of Equations 6.7 and 6.4 shows that the recoil effect introduces the factor $E'/E$.

5) Spin-$\frac{1}{2}$ Electron Scattering off a Spin-$\frac{1}{2}$ Point Particle which can Recoil

This simply corresponds to $e^+\mu^- \rightarrow e^+\mu^-$ scattering which we worked out in Chapter 5. Recall Equation 5.82
\[
\left( \frac{d\sigma}{d\Omega} \right) = \left( \frac{\alpha^2}{4E^2 \sin^4 \theta / 2} \right) \left( \frac{E'}{E} \right) \left( \cos^2 \theta - \frac{q^2}{2M^2} \sin^2 \theta \right) \tag{6.8}
\]

A comparison of Equations 6.8 and 6.7 shows that the spin-\(\frac{1}{2}\) target leads to an additional term, \(-\frac{q^2}{2M^2} \sin^2 \theta / 2\), which represents a magnetic interaction. The spin of the electron can therefore be flipped via the magnetic interaction. Equation 6.8 shows that electron can now scatter to 180°.

We are now ready to consider \(ep\) elastic scattering. The proton is no longer considered as a structureless point particle. Instead, proton is a spin-\(\frac{1}{2}\) composite particle. The \(ep\) elastic scattering can be represented by the following diagram:

The transition matrix \(T_{fi}\) can be written as (Equation 5.17)

\[
T_{fi} = -i \int j_\mu \left( -\frac{1}{q^2} \right) J^\mu d^4 x \tag{6.9}
\]

The current from the point-like electron is

\[
j_\mu = -ie\bar{u}(K')\gamma_\mu u(K)e^{i(K'-K)\cdot x} \tag{6.10}
\]

For the proton, the current can be written analogously as

\[
J^\mu = e\bar{u}(p')[\quad ]u(p)e^{i(p'-p)\cdot x} \tag{6.11}
\]

Equation 6.11 reflects the fact that initial and final states of the hadron are protons (elastic scattering). However, the 4-vector represented by [ ] is no longer \(\gamma^\mu\).
One can write down a most general expression for the 4-vector, constructed out of the $\gamma$-matrices and the 4-momenta, $p$ and $p'$, as

$$
\left[ K_1 \gamma^\mu + K_2 i\sigma^{\mu\nu} (p' - p)_\nu + K_3 i\sigma^{\mu\nu} (p' + p)_\nu \\
+ K_4 (p' - p)^\mu + K_5 (p' + p)^\mu \right]
$$

(6.12)

Now, the “Gordon decomposition” can be used to express $(p' + p)^\mu$ in terms of $\gamma^\mu$ and $\sigma^{\mu\nu} (p' - p)_\nu$:

$$
\bar{u}(p') \gamma^\mu u(p) = \frac{1}{2m} \bar{u}(p') \left[ (p' + p)^\mu + i\sigma^{\mu\nu} (p' - p)_\nu \right] u(p)
$$

(6.13)

Using Equation 6.13, the $K_5$ term can be expressed in terms of the $K_1$ and $K_2$ terms. Furthermore, it can be shown that

$$
i \bar{u}(p') i\sigma^{\mu\nu} (p' + p)_\nu u(p) = -\bar{u}(p')(p' - p)^\mu u(p)
$$

(6.14)

using the fact that $(p' - m)u(p) = 0$ and $\bar{u}(p')(p' - m) = 0$. Therefore, the $K_3$ term can be expressed in terms of the $K_4$ term. The $p' + p$ terms in Equation 6.12 are not independent of the other three terms.

A general expression for the hadronic current for proton in the $ep$ scattering is

$$
J^\mu = e\bar{u}(p') \left[ F_1(q^2) \gamma^\mu + \frac{\gamma^\mu}{2M} F_2(q^2) i\gamma^{\mu\nu} q_\nu + F_3(q^2) q^\mu \right] u(p) e^{i(p' - p)\cdot x}
$$

(6.15)

Current conservation, $\partial_\mu J^\mu = 0$, implies that $q_\mu J^\mu = 0$ and we have

$$
F_1(q^2) \bar{u}(p')(p' - p)u(p) + F_2(q^2) \bar{u}(p') \frac{\gamma^\mu}{2m} q_\mu \sigma^{\mu\nu} q_\nu u(p) \\
+ F_3(q^2) \bar{u}(p') q^2 u(p) = 0
$$

(6.16)

The first term in Equation 6.16 vanishes since $u(p), \bar{u}(p')$ satisfy the Dirac Equation. The second term is also equal to zero since $\sigma^{\mu\nu}$ is antisymmetric with
respect to $\mu\nu$ exchange. Therefore, $F_3(q^2) = 0$, and the hadronic current can be written as

$$J^\mu = e\bar{u}(p') \left[ F_1(q^2)\gamma^\mu + \frac{\kappa}{2M} F_2(q^2)i\sigma^{\mu\nu}q_\nu \right] u(p)e^{i(p' - p)\cdot x}$$  \hspace{1cm} (6.17)$$

Equation 6.17 can be re-expressed - using the Gordon decomposition (Equation 6.13) – as

$$J^\mu = e\bar{u}(p') \left[ F_1(q^2) + \kappa F_2(q^2) \right] \gamma^\mu - \frac{\kappa F_2(q^2)}{2M}(p' + p)^\mu \right] u(p)e^{iq}$$  \hspace{1cm} (6.18)$$

For a point-like proton, $\kappa = 0$ and $F_1(q^2) = 1$. $\kappa$ is the anomalous magnetic moment produced by the motion of the constituents. Note that $F_1, F_2$ are functions of $q^2$ only. Other Lorentz scalars, such as $p\cdot q$, can be expressed in terms of $q^2$.

To evaluate the $ep$ elastic scattering cross section, we have

$$|M|^2 = \frac{e^4}{q^4} L_{\mu\nu} w^{\mu\nu}$$

where

$$w^{\mu\nu} = \frac{1}{2} \sum_{\text{spin}} \bar{u} \left[ (F_1 + \kappa F_2)\gamma^\mu - \frac{\kappa F_2}{2M}(p' + p)^\mu \right] u$$

\hspace{1cm} (6.19)$$

$$= \left[ \bar{u} \left[ (F_1 + \kappa F_2)\gamma^\nu - \frac{\kappa F_2}{2M}(p' + p)^\nu \right] u \right]^\dagger$$

and $L_{\mu\nu}$ is the leptonic tensor for the electrons (Equation 5.73).

Using standard techniques for calculating the traces, the three terms in $w^{\mu\nu}$ are:

1) $$(F_1 + \kappa F_2)^2 \text{tr} \left[ (p' + M)\gamma^\mu (p' + M)\gamma^\nu \right]$$

= $4(F_1 + \kappa F_2)^2 \left[ p'^\mu p'^\nu + p'^\nu p'^\mu - g^{\mu\nu}(p\cdot p') + g^{\mu\nu}M^2 \right]$

2) $$-\frac{\kappa F_2}{2M}(F_1 + \kappa F_2)\text{tr} \left[ (p' + M)\gamma^\mu (p' + M)(p + p')^\nu \right]$$

= $-2\kappa F_2(F_1 + \kappa F_2)^2 (p + p')^\mu (p + p')^\nu$

3) $$-\frac{\kappa^2 F_2^2}{2M}(F_1 + \kappa F_2)(p - p')^\mu (p + p')^\nu$$

= $-\kappa^2 F_2^2 (F_1 + \kappa F_2)^2 (p - p')^\mu (p + p')^\nu$
$$-\frac{\kappa^2 F_2^2}{4M^2} \text{tr} \left[ \left( \not{p} + M \right) \left( p + p' \right) \mu \left( \not{p} + M \right) \left( p + p' \right) \nu \right]$$

$$= -\frac{\kappa^2 F_2^2}{4M^2} \left( p + p' \right)^2 \left( p + p' \right) \mu \left( p + p' \right) \nu$$

The \( ep \) elastic scattering cross section becomes

$$\frac{d\sigma}{d\Omega_{\text{lab}}} = \left( \frac{\alpha^2}{4E^2 \sin^4 \theta / 2} \right) \frac{E'}{E} \left( F_1^2 - \frac{\kappa^2 q^2}{4M^2} F_2^2 \right) \cos^2 \frac{\theta}{2}$$

$$- \frac{q^2}{2M^2} \left( F_1 + \kappa F_2 \right)^2 \sin^2 \frac{\theta}{2} \right)$$

(6.20)

It is conventional to introduce the ‘Sachs’ form factors \( G_E \) and \( G_M \):

$$G_E = F_1 + \frac{\kappa q^2}{4M^2} F_2 \quad \text{(electric form factor)}$$

$$G_M = F_1 + \kappa F_2 \quad \text{(magnetic form factor)}$$

(6.21)

and the \( ep \) elastic cross section is written as

$$\frac{d\sigma}{d\Omega_{\text{lab}}} = \left( \frac{\alpha^2}{4E^2 \sin^4 \theta / 2} \right) \frac{E'}{E} \left( \frac{G_E^2 + \tau G_M^2}{1 + \tau} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right)$$

(6.22)

where \( \tau = -\frac{q^2}{4M^2} \).

For \( q^2 \to 0 \), the electron beam effectively sees a proton with charge \( e \) and magnetic moment of \( (1 + \kappa) \frac{e}{2M} \). We therefore have, from Equation 6.17,

$$F_1^p(0) = 1 \quad F_2^p(0) = 1$$

(6.23)

Similarly, for electron scattering off a neutron,
\[ F_1^n(0) = 0 \quad F_2^n(0) = 1 \quad (6.24) \]

Equation 6.21 gives
\[
\begin{align*}
G_E^p(0) &= 1 \\
G_E^n(0) &= 0 \\
G_M^p(0) &= 1 + \kappa_p = \mu_p = 2.79 \\
G_M^n(0) &= \kappa_n = \mu_n = -1.91
\end{align*}
\quad (6.25)\]

This shows that
\[
G_E^p(0) = \frac{G_M^p(0)}{\mu_p} \quad (6.26)
\]

It is found experimentally that \( G_E^p(Q^2) \) and \( G_M^p(Q^2) \) have very similar \( Q^2 \) dependence, and
\[
G_E^p(Q^2) \approx \frac{G_M^p(Q^2)}{\mu_p} = \left(1 + \frac{Q^2}{0.71 \text{ GeV}^2}\right)^{-2}
\quad (6.27)
\]

Recent data from Jefferson Lab showed, however, that \( \mu_p G_E^p(Q^2)/G_M^p(Q^2) \) starts to deviate from unity as \( Q^2 \) becomes large. The root-mean-square radius of the proton is determined to be
\[
\left< r^2 \right> = -6 \frac{dG(Q^2)}{dQ^2} \bigg|_{Q^2 \rightarrow 0} = 0.66 \text{ fm}^2
\quad (6.28)
\]
\[
\left< r^2 \right>^{1/2} = 0.81 \text{ fm}
\]