Chapter 4

DIRAC EQUATION

The non-relativistic Schrödinger equation was obtained by noting that the Hamiltonian

\[ H = \frac{P^2}{2m} \]  

(1)

can be transformed into an operator form with the substitutions

\[ H \rightarrow i \frac{\partial}{\partial t} \quad (\hbar = 1, \ c = 1) \]  

\[ P \rightarrow -i \vec{\nabla} \]  

(2)

resulting

\[ i \frac{\partial}{\partial t} \psi(x, t) = \frac{\nabla^2}{2m} \psi(x, t) \]  

(3)

This equation is linear in the time-derivative and quadratic in the space derivative, and is manifestly non-covariant under Lorentz transformation. Clearly, the left-hand side and the right-hand side of Equation 3 transforms differently under Lorentz transformation.

We briefly review Lorentz transformation.

Space-time coordinates \((t, x, y, z)\) are denoted by the contravariant 4-vector

\[ x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) \]

The covariant 4-vector, \(x_\mu\), is

\[ x_\mu = (x_0, x_1, x_2, x_3) = (t, -x, -y, -z) = g_{\mu \nu} x^\nu \]

\[ g_{\mu \nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \]  

(4)

\(g_{\mu \nu}\) is the metric tensor
The Lorentz transformation relates $x^\mu$ to $x'^\mu$:

$$\left( x^\mu \right)' = \Lambda^\mu_\nu x^\nu \tag{5}$$

For an inertial system $S'$ moving along $z$ with $V/c = \beta$, we have

$$\Lambda = \begin{bmatrix}
\gamma & 0 & 0 & -\beta \gamma \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\beta \gamma & 0 & 0 & \gamma \\
\end{bmatrix} \tag{6}$$

where $\gamma = \left(1 - \beta^2\right)^{-1/2}$

Any set of four quantities which transform like $x^\mu$ under Lorentz transformation is called a four-vector. The total energy $E$ and the momentum $\vec{P}$ form a four-vector

$$P^\mu = (P^0, P^1, P^2, P^3) = (E, \vec{P}) \tag{7}$$

Scalar product of two 4-vectors are invariant under Lorentz transformation

$$A_\mu B^\mu = g_\mu_\nu A^\nu B^\mu \tag{8}$$

Some examples:

$$x \cdot x = x_\mu x^\mu = t^2 - x^2$$

$$P \cdot P = P_\mu P^\mu = E^2 - P^2$$

$$x \cdot P = x_\mu P^\mu = t(E - \vec{x} \cdot \vec{P})$$

The $H \rightarrow i \frac{\partial}{\partial t}$, $\vec{P} \rightarrow -i \vec{\nabla}$ transcription is Lorentz covariant

$$P^\mu \rightarrow i \frac{\partial}{\partial x^\mu} = i \partial^\mu$$

One can readily obtain the continuity equation from the Schrödinger equation:
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \tag{9}
\]

where \( \rho = \psi^* \psi \geq 0, \quad \vec{J} = -\frac{i}{2m}(\psi^* \nabla \psi - \psi \nabla \psi^*) \)

\( \rho \) is the probability density, and \( \vec{J} \) is the current density.

Note that the continuity equation can be written in a covariant form

\[
\partial \rho^\mu = 0, \quad j^\mu = (\rho, \vec{J}) \tag{10}
\]

The early attempt for formulating a Lorentz covariant equation started from the relativistic energy-momentum relation

\[
E^2 = P^2 + m^2
\]

with the \( E \rightarrow i \frac{\partial}{\partial t}, \quad \vec{P} \rightarrow -i\nabla \) transcription, one obtains

\[
-\frac{\partial^2}{\partial t^2} \psi = (-\nabla^2 + m^2)\psi \tag{11}
\]

which is the relativistic Schrödinger equation, or the Klein-Gordon equation.

Equation 11 can be written as

\[
(\Box^2 + m^2)\psi = 0 \tag{12}
\]

where

\[
\Box^2 = \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} = \partial_{\mu} \partial^{\mu} \tag{13}
\]

is the d’Alembertian operator, a Lorentz invariant operator.

From the Klein-Gordon equation, one can readily derive the following continuity equation:
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \]

\[ \rho = i \left( \phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) \]

\[ j^{\mu} = i \left( \phi^* \partial^{\mu} \phi - \phi \partial^{\mu} \phi^* \right) \]

\[ \vec{J} = -i \left( \phi^* \nabla \phi - \phi \nabla \phi^* \right) \]

For a plane-wave solution to Klein-Gordon equation

\[ \phi = Ne^{-i(Et - \vec{p} \cdot \vec{x})} \]

We have

\[ \rho = 2E |N|^2, \quad \vec{J} = 2\vec{P} |N|^2 \]

Since \( E = \pm \left( P^2 + m^2 \right)^{1/2} \), \( \rho \) can be negative for the \( E < 0 \) solutions.

The origin of this problem is the second derivative in time for the term \( \partial^2 / \partial t^2 \).

Dirac tried to find a relativistic covariant equation with a positive-definite probability density. He assumed an equation linear in \( \frac{\partial}{\partial t} \) and in \( \nabla \):

\[ H\psi = \left( \vec{\alpha} \cdot \vec{P} + \beta m \right) \psi \]

\[ \vec{\alpha} \cdot \vec{P} = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 \]

It is evident that \( \alpha \)'s and \( \beta \) cannot be just numbers, because the equation would not be invariant even under spatial rotation. Dirac proposed that the equation be considered as a matrix equation. It should satisfy \( H^2 \psi = \left( P^2 + m^2 \right) \psi \).

Applying \( H \) to Equation 18 leads to

\[ H^2 \psi = \left[ \sum_i \alpha_i^2 P_i^2 + \sum_{i \neq j} (\alpha_i \alpha_j + \alpha_j \alpha_i) P_i P_j + \sum_i (\alpha_i \beta + \beta \alpha_i) m P_i + \beta^2 m^2 \right] \psi \]
Therefore
\[ \{ \alpha_i, \alpha_j \} = 2\delta_{ij} ; \{ \alpha_i, \beta \} = 0 \]
\[ \alpha_i^2 = \beta^2 = 1 \]

What are the other properties of \( \alpha_i, \beta \)?

First, \( \alpha_i, \beta \) are Hermitian, since \( H \) is Hermitian (see Equation 18).

In addition, \( \text{Tr}(\alpha_i) = 0 \), \( \text{Tr}(\beta) = 0 \). This can be shown by the anticommutator relations in Equation 19. Recall \( \text{Tr}(AB) = \text{Tr}(BA) \)

\[ \text{Tr}(\alpha_i) = \text{Tr}(\alpha_i\beta^2) = \text{Tr}(\beta\alpha_i\beta) = \text{Tr}(-\alpha_i\beta\beta) = -\text{Tr}(\alpha_i) \Rightarrow \text{Tr}(\alpha_i) = 0 \tag{20} \]

Similarly, \( \text{Tr}(\beta) = 0 \). Therefore, \( \alpha_i, \beta \) are traceless, Hermitian matrices.

The eigenvalues of \( \alpha_i, \beta \) can only be +1 or -1 since \( \alpha_i^2 = \beta^2 = 1 \).

\[ \alpha_i x = \lambda x \Rightarrow \alpha_i^2 x = \lambda^2 x = x \Rightarrow \lambda = \pm 1 \tag{21} \]

The trace of a matrix is equal to the sum of all its eigenvalues. Therefore, for a traceless matrix having +1, -1 as possible eigenvalues, it must have an even number of dimension \( n \). \( n \) cannot be 2, since only three traceless \( n = 2 \) independent matrices exist (\( \sigma_x, \sigma_y, \sigma_z \), for example), and we need four traceless matrices to accommodate \( \alpha_i, \alpha_2, \alpha_3, \beta \).

\( n = 4 \) is the smallest possible dimension for \( \alpha_i, \beta \). The Pauli-Dirac representation is given as

\[ \alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \quad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{22} \]

The Dirac equation \( \left[ \bar{\alpha} \cdot \bar{P} + m\beta \right] \psi = H\psi \) can be expressed in a more covariant fashion by multiplying \( \beta \) from the left:

\[ (\beta H - (\beta \bar{\alpha}) \cdot \bar{P} - m) \psi = 0 \tag{23} \]
Introducing \( \gamma^\mu = (\beta, \beta \bar{\alpha}) \), Equation 23 becomes

\[
\left( \gamma^\mu P_\mu - m \right) \psi = 0
\]

(24)

\( P_\mu = i \partial_\mu \), therefore

\[
\left( i \gamma^\mu \partial_\mu - m \right) \psi = 0 \quad \text{(recall } \partial_\mu = \frac{\partial}{\partial x^\mu})
\]

(25)

Explicitly,

\[
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}
\]

(26)

\( \gamma^\mu \)'s satisfy the anticommutation relation

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu \nu} I
\]

\[
(\gamma^0)^2 = I, \quad (\gamma^i)^2 = -I
\]

(27)

Although \( \gamma^0 \) is Hermitian, \((\gamma^0)^+ = \gamma^0\), \( \gamma^i \) is anti-Hermitian:

\[(\gamma^i)^+ = -\gamma^i \]

(28)

However,

\[(\gamma^\mu)^+ = \gamma^0 \gamma^\mu \gamma^0 \quad \text{and} \quad \gamma^\mu = \gamma^0 (\gamma^\mu)^+ \gamma^0
\]

(29)

To obtain the adjoint Dirac equation, one takes Hermitian conjugate of Equation 25

\[
-i \partial_\mu \psi^+ (\gamma^\mu)^+ - m \psi^+ = 0
\]

(30)

Since \( \gamma^\mu \) is not Hermitian for \( \mu = 1, 2, 3 \), it is useful to multiply Equation 30 from the right by \( \gamma^0 \) to yield (using Equation 29)

\[
-i \partial_\mu (\psi^+ \gamma^0) \gamma^\mu - m (\psi^+ \gamma^0) = 0
\]

(31)

Define

\[
\bar{\psi} = \psi^+ \gamma^0
\]

(32)

the adjoint Dirac equation is

\[
i \partial_\mu \bar{\psi} \gamma^\mu + m \bar{\psi} = 0
\]

(33)
or
\[ \bar{\psi} \left( \not{P} + m \right) = 0 \] (34)

where
\[ \not{P} = \gamma^\mu P_\mu \]

and \( P \) acts to its left in Equation 34.

One is now ready to check the continuity equation and the probability density and current density for the Dirac equation.

Multiplying Equation 25 by \( \bar{\psi} \) to its left, and Equation 33 by \( \psi \) to its right, we obtain
\[
\begin{align*}
i\bar{\psi}\gamma^\mu \partial_\mu \psi - \bar{\psi} m \psi &= 0 \\
i\partial_\mu \bar{\psi} \gamma^\mu \psi + m \bar{\psi} \psi &= 0
\end{align*}
\] (35)

Adding these two equations yields
\[ \partial_\mu \left( \bar{\psi} \gamma^\mu \psi \right) = 0 \] (36)

Define
\[ j^\mu = \bar{\psi} \gamma^\mu \psi \] (37)
the continuity equation \( \partial_\mu j^\mu = 0 \) follows.

Note that
\[ \rho = j^0 = \bar{\psi} \gamma^0 \psi = \psi^+ \psi = \sum_{i=1}^{4} |\psi_i|^2 \geq 0 \]

Hence the probability density is indeed positive definite for the Dirac equation.

As will be shown later \( j^\mu \) transforms like a 4-vector, and \( \rho = j^0 \) is not a Lorentz invariant. Rather, it is the time component of a 4-vector. It is not too surprising that \( \rho \) is not a Lorentz invariant, since a probability density satisfies the following:
\[ \int \rho d^3x = \text{const} \] (38)

Since \( d^3x \) undergoes Lorentz contraction, it is natural that \( \rho \) must also change under the Lorentz transformation in order to preserve the integral.
Each component of the four Dirac wave function components satisfies the Klein-Gordon equation. Multiplying Equation 25 from the left by \(i\gamma^\mu \partial_\mu\), we have

\[
-\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \psi - m^2 \psi = 0
\]  

or

\[
\left[ \frac{1}{2} \left( \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \right) \partial_\mu \partial_\nu + m^2 \right] \psi = 0
\]

hence

\[
\left( g^{\mu\nu} \partial_\mu \partial_\nu + m^2 \right) \psi = 0
\]

or

\[
\left( \partial_\mu \partial^\mu + m^2 \right) \psi = 0
\]  

(39)

(40)

Note that Equation 40 is understood as four separate uncoupled equations for each component of \(\psi\).

We turn next to the solutions to the Dirac equation for a free particle.

Because of Equation 40, the Dirac equation admits a free-particle solution as follows

\[
\psi = u(P) e^{-ip\cdot x}
\]  

(41)

where \(u(P)\) is independent of \(x\) and is a 4-component spinor. Substituting Equation 41 into Equation 25 gives

\[
\left( \gamma^\mu P_\mu - m \right) u(P) = 0, \quad \text{or} \quad \left( \not{p} - m \right) u(P) = 0
\]  

(42)

Eq. 42 is the Dirac equation in the momentum space. Note that \(P_\mu\) in Eq. 42 is no longer a differential operator (just a multiplicative operator). We now simply write \(u(P)\) as \(u\). Similarly, the adjoint equation is readily obtained:

\[
\bar{u} \left( \not{p} - m \right) = 0
\]  

(43)

We now try to find solutions to the Dirac Equation. For a Dirac particle at rest, \(\not{p} = 0\), and Equation 42 becomes
\[ (\gamma^0 E - m) u = 0 \quad (\psi = u e^{-iEt}) \] (44)

or

\[ E u = m \gamma^0 u = \begin{pmatrix} m \\ m \\ -m \\ -m \end{pmatrix} u \] (45)

The eigenvectors are

\[
\begin{align*}
    u_1 &= N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
    u_2 &= N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
    u_3 &= N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
    u_4 &= N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{align*}
\] (46)

with eigenvalues \( m, m, -m, -m \), respectively. \( N \) is the normalization factor.

To obtain the solutions for \( \vec{p} \neq 0 \), one can either solve the Dirac Equation directly, or by boosting the \( \psi (p = 0) \) solution to another inertia frame moving with \( -\vec{p} \) with respect to the rest frame.

It is instructive to examine the method involving the Lorentz boost. To use this method, we need to first address the question – How does \( \psi \) transform under Lorentz transformation? Let us suppose that in reference frame \( S \), the Dirac equation is written as

\[
\left( i \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi (x) = 0
\] (47)

In another frame \( S' \), where \( x'^\mu = \Lambda^\mu_\nu x^\nu \), the Dirac equation becomes

\[
\left( i \gamma^\mu \frac{\partial}{\partial x'^\mu} - m \right) \psi' (x') = 0
\] (48)

Now,

\[
\psi' (x') = \psi' (\Lambda x) = S \psi (x)
\] (49)
and
\[ \psi(x) = S^{-1}\psi'(x') \] (50)

Substituting Equation 50 into Equation 47 and noting
\[ \frac{\partial}{\partial x'^\nu} = \frac{\partial}{\partial x^\nu} = \Lambda^\nu_{\mu} \frac{\partial}{\partial x'^\nu} \] (51)

Equation 47 becomes, upon multiplying \( S \) from the left,
\[ \left( iS\gamma^\mu \Lambda^\nu_{\mu} \frac{\partial}{\partial x'^\nu} S^{-1} - m \right) \psi'(x') = 0 \] (52)

Requiring Equation 52 to be identical to Equation 48 gives
\[ iS\gamma^\mu \Lambda^\nu_{\mu} \frac{\partial}{\partial x'^\nu} S^{-1} = i\gamma^\mu \frac{\partial}{\partial x'^\mu} \]

or
\[ S\gamma^\mu \Lambda^\nu_{\mu} S^{-1} = \gamma^\nu \]

Equivalently,
\[ \Lambda^\nu_{\mu} \gamma^\mu = S^{-1} \gamma^\nu S \]

Our task is to find \( S \), which is a 4 x 4 matrix, for any Lorentz transformation \( \Lambda \), such that Equation 53 is satisfied.

We consider three cases of Lorentz transformation:

1) Lorentz boost
2) Spatial rotation
3) Space inversion (parity operation)

The cases 1) and 2) can be built up with infinitesimal Lorentz transformations.

Consider an infinitesimal Lorentz transformation
\[ \Lambda^\nu_{\mu} = \delta^\nu_{\mu} + \Delta \omega^\nu_{\mu} = \delta^\nu_{\mu} + \Delta \omega(I_n)^{\nu}_{\mu} \] (54)
\( \Delta \omega \) is an infinitesimal quantity, and \( I_\mu \) is a 4 x 4 matrix corresponding to a specific Lorentz transformation.

For a transformation to \( S' \) moving along the x-axis with an infinitesimal velocity \( \Delta \omega = \Delta \beta \), we have

\[
I^\nu_\mu = \begin{bmatrix}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \quad I^{\nu\mu} = \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \tag{55}
\]

Similarly, for a rotation through an angle \( \Delta \omega \) around the z-axis, the corresponding \( I^\nu_\mu \) is

\[
I^\nu_\mu = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & +1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \quad I^{\nu\mu} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \tag{56}
\]

In order to satisfy the relation for Lorentz transformation

\[
\Lambda^\nu_\mu \Lambda^\mu_\sigma = \delta^\nu_\sigma \tag{57}
\]

the infinitesimal transformation, Equation 54, satisfies

\[
I^{\mu\nu} = -I^{\nu\mu} \tag{58}
\]

as can be readily verified for Equations 55 and 56.

There are 6 independent \( I^{\mu\nu} \), 3 for boosts and 3 for rotations.

Expand \( S \) in powers of \( \Delta \omega \):

\[
S = 1 - \frac{i}{4} \sigma_{\mu\nu} \Delta \omega^{\mu\nu} \tag{59}
\]

\( \sigma_{\mu\nu} \) is the generator for the corresponding infinitesimal Lorentz transformation.
Inserting Equation 59 and Equation 54 into Equation 53, we have

\[ 2i\left[ g^\gamma_\alpha \gamma_\beta - g^\nu_\beta \gamma_\alpha \right] = \left[ \gamma^\nu, \sigma_{\alpha\kappa} \right] \]  

which is satisfied by

\[ \sigma_{\mu\nu} = \frac{i}{2} \left[ \gamma_\mu, \gamma_\nu \right] \]  

The anti-symmetric tensor \( \sigma_{\mu\nu} \) has the following non-vanishing elements

\[ \sigma_{oi} = -\sigma_{io} = -i\alpha_i \]  

for boost and

\[ \sigma_{ij} = -\sigma_{ji} = \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix} \]  

for rotation \( ijk \) cyclic.

We are ready to construct a finite spinor transformation \( S \) from successive infinitesimal transformation

\[ S = \lim_{N \to \infty} \left( 1 - \frac{i}{4N} \sigma_{\mu\nu} I^\mu_\nu \right)^N \]  

To prove Equation 60

\[ S = 1 - \frac{i}{4} \sigma_{\alpha\beta} \Delta \omega^{\alpha\beta} \]  

\[ S^+ = 1 + \frac{i}{4} \sigma_{\alpha\beta} \Delta \omega^{\alpha\beta} \]  

\[ \Lambda^\nu_\mu = \delta^\nu_\mu + \Delta \omega^\nu_\mu \]  

Since

\[ \Lambda^\nu_\mu \gamma^\mu = S^{-1} \gamma^\nu S \]  

We have

\[ \left( \delta^\nu_\mu + \Delta \omega^\nu_\mu \right) \gamma^\mu = \left( 1 + \frac{i}{4} \sigma_{\alpha\beta} \Delta \omega^{\alpha\beta} \right) \gamma^\nu \left( 1 - \frac{i}{4} \sigma_{\alpha\beta} \Delta \omega^{\alpha\beta} \right) \]
or
\[ \gamma^\nu + \Delta \omega^\nu_\mu \gamma^\mu = \gamma^\nu - \frac{i}{4} \gamma^\nu \sigma_{\alpha\beta} \Delta \omega^{\alpha\beta} + \frac{i}{4} \sigma_{\alpha\beta} \Delta \omega^{\alpha\beta} \gamma^\nu \]
(ignoring \((\Delta \omega)^2\) term)

Therefore
\[ \Delta \omega^\nu_\mu \gamma^\mu = -\frac{i}{4} \Delta \omega^{\alpha\beta} \left[ \gamma^\nu, \sigma_{\alpha\beta} \right] \quad (a) \]

but
\[ \Delta \omega^\nu_\mu \gamma^\mu = \Delta \omega^\nu_\nu \gamma^\nu = \frac{1}{2} \left[ \Delta \omega^{\nu\beta} \gamma^\beta + \Delta \omega^{\nu\alpha} \gamma^\alpha \right] \]
\[ = \frac{1}{2} \left[ \Delta \omega^{\nu\beta} \gamma^\beta - \Delta \omega^{\nu\alpha} \gamma^\alpha \right] \]
\[ = \frac{1}{2} \left[ g^{\nu\alpha} \gamma^\alpha - g^{\nu\beta} \gamma^\beta \right] = \left[ \gamma^\nu, \sigma_{\alpha\beta} \right] \quad (b) \]

Inserting (b) into (a), we obtain
\[ 2i \left[ g^{\nu\alpha} \gamma^\alpha - g^{\nu\beta} \gamma^\beta \right] = \left[ \gamma^\nu, \sigma_{\alpha\beta} \right] \]

For a Lorentz boost along the \(x\)-axis,
\[ \psi'(x') = \exp \left( -\frac{i}{2} \omega \sigma_{01} \right) \psi(x) \]
\[ = \exp \left( -\frac{\omega}{2} \alpha_1 \right) \psi(x) \quad (65) \]

where \( \tanh \omega = \beta \), \( \cosh \omega = \frac{1}{\sqrt{1 - \beta^2}} = \gamma \), \( \sinh \omega = \beta \gamma \)

Similarly, for a rotation around the \(z\)-axis of an angle \(\omega\):
\[ \psi'(x') = \exp \left( \frac{i}{2} \omega \sigma^{12} \right) \psi(x) \quad (66) \]

where
\[ \sigma^{12} = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \]

\[ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]
Equation 66 is similar to the rotation of two-component Pauli spinor

\[ \psi'(x') = e^{\frac{i}{2} \sigma \cdot \vec{\sigma}} \psi(x) \]  \hspace{1cm} (67)

Are the spinor transformation \( S \) unitary?

For spatial rotation, \( S = S_{Rot} \), \( S \) is unitary, since

\[ S_{Rot}^* = e^{-\frac{i}{2} \sigma \cdot \vec{\sigma}} = e^{-\frac{i}{2} \sigma_i \sigma_i} = S^{-1}_{Rot} \]  \hspace{1cm} (68)

but for Lorentz boost

\[ S_{Lor}^* = e^{-\frac{i}{2} \alpha \cdot \vec{\alpha}} = e^{-\frac{i}{2} \alpha_i \alpha_i} = S_{Lor} \neq S^{-1}_{Lor} \]  \hspace{1cm} (69)

Nevertheless, both \( S_{Rot} \) and \( S_{Lor} \) have the property

\[ S^{-1} = \gamma_o \cdot S^+ \cdot \gamma_o \]  \hspace{1cm} (70)

which can be verified by expanding \( S_{Rot}, S_{Lor} \) in power series.

Here is proof for \( \beta = \tanh \omega \)

Consider a Lorentz boost along the \( x \)-axis

\[ I = I^\nu_\mu = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ I^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad I^3 = I \]

\[ \chi^\nu = \lim_{N \to \infty} \left( 1 + \frac{\omega I}{N} \right)^\nu_\mu \chi^\mu \]
\[ (e^{\omega I})_{\mu}^{\nu} x^\mu = (\cosh \omega I + \sinh \omega I)_{\mu}^{\nu} x^\mu = (1 - I^2 + I^2 \cosh \omega + I \sinh \omega)_{\mu}^{\nu} x^\mu \]

\[
x' = \begin{bmatrix} \cosh \omega & -\sinh \omega & 0 & 0 \\ -\sinh \omega & \cosh \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x,
\]
\[ t' = (\cosh \omega) t - (\sinh \omega) x \]
\[ x' = -(\sinh \omega) t + (\cosh \omega) x \]

We also have
\[ t' = \gamma \left( t - \beta x \right) \]
\[ x' = \gamma \left( -\beta t + x \right) \]

Therefore
\[ \gamma = \cosh \omega \quad \gamma \beta = \sinh \omega \quad \beta = \tanh \omega \]

The fact that \( S_{Lor} \) is not unitary (Equation 69) should not be too surprising. This simply reflects the situation that \( \psi^+ \psi \) is not a conserved quantity in a Lorentz boost.

We now consider space inversion:
\[ \Lambda_{\mu}^{\nu} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \]

This discrete transformation cannot be constructed out of infinitesimal transformation. Rather, we rely on the definition of \( S \) (Equation 53) to find the parity transformation \( S_p \) for the Dirac spinor \( \psi \).

Equation 53 and Equation 71 imply
\[ S_p^{-1} \gamma^\nu S_p = \gamma^\nu \]
\[ S_p^{-1} \gamma^i S_p = -\gamma^i \]
This can be satisfied with

\[
S_p = \gamma^o = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

Note that \(S_p\) also satisfies Equation 70 since

\[
S_p^{-1} = \gamma^o S_p^+ \gamma^o
\]

For a Dirac particle at rest, Equation 73 shows that \(U_1\) and \(U_2\) have positive parity while \(U_3\) and \(U_4\) have negative parity (Equation 46)!

Having established the properties of \(S\), we can show that \(\bar{\psi} \psi\) transforms like a scalar, while \(\bar{\psi} \gamma^\mu \psi\) transforms like a 4-vector under Lorentz transformation.

\[
\bar{\psi}'(x') = \psi'(x')^+ \gamma^o = (S \psi(x))^+ \gamma^o = \psi(x)^+ S^+ \gamma^o = \psi(x)^+ S^{-1} = \bar{\psi}(x) S^{-1}
\]

Therefore

\[
\bar{\psi}'(x') \psi'(x') = \bar{\psi}(x) S^{-1} S \psi(x) = \bar{\psi}(x) \psi(x)
\]

\[
\Rightarrow \bar{\psi} \psi\text{ is a scalar}
\]

\[
\bar{\psi}'(x') \gamma^\mu \psi'(x') = \bar{\psi}(x) S^{-1} \gamma^\mu S \psi(x)
\]

\[
= \bar{\psi}(x) \Lambda^\mu \gamma^o \psi(x)
\]

\[
= \Lambda^\mu \left( \bar{\psi}(x) \gamma^o \psi(x) \right)
\]

\[
\Rightarrow \bar{\psi} \gamma^\mu \psi\text{ is a 4-vector}
\]
To obtain the plane-wave solution to the Dirac equation for a particle moving along the $x$-axis with velocity $\beta$, we boost along $-x$ with a velocity of $-\beta$.

\[ S = e^{\frac{i}{2}\omega \sigma_0} \sigma_0 = -i \alpha_1 \]
\[ = e^{-\omega \sigma_0} = \cosh \frac{\omega}{2} - \alpha_1 \sinh \frac{\omega}{2} \]
where $\tanh \omega = -\beta$

The spinors $u'(p)$ are
\[
u'(p) = \begin{bmatrix} \cosh \left( \frac{\omega}{2} \right) - \alpha_1 \sinh \left( \frac{\omega}{2} \right) \end{bmatrix} U'(0)
= \begin{bmatrix} 1 & 0 & 0 & -\tanh \frac{\omega}{2} \\ 0 & 1 & -\tanh \frac{\omega}{2} & 0 \\ 0 & -\tanh \frac{\omega}{2} & 1 & 0 \\ -\tanh \frac{\omega}{2} & 0 & 0 & 1 \end{bmatrix} u'(0)
\] (77)

where
\[
\tanh \left( \frac{\omega}{2} \right) = \frac{\tanh \omega}{1 + \sqrt{1 - \tanh^2 \omega}} = \frac{-\beta}{1 + \sqrt{1 - \beta^2}}^{\frac{1}{2}}
= \frac{-\beta}{1 + \frac{1}{\gamma}} = \frac{-\beta \gamma}{1 + \gamma} = \frac{-\beta \gamma m}{(1 + \gamma)m} = \frac{-P}{E + m}
\] (78)

and
\[
\cosh \left( \frac{\omega}{2} \right) = \frac{1}{\sqrt{\left( 1 - \tanh^2 \left( \frac{\omega}{2} \right) \right)^{\frac{1}{2}}}} = \frac{1}{\sqrt{\left( \frac{2}{1 + \gamma} \right)^{\frac{1}{2}}}} = \sqrt{\frac{m + E}{2m}}
\]

using Equation 78, Equation 77 becomes
\[ u^\nu(p) = \sqrt{\frac{m + E}{2m}} \begin{bmatrix} 1 & 0 & 0 & \frac{P_x}{E + m} \\ 0 & 1 & \frac{P_x}{E + m} & 0 \\ 0 & \frac{P_x}{E + m} & 1 & 0 \\ \frac{P_x}{E + m} & 0 & 0 & 1 \end{bmatrix} u^\nu(0) \] (79)

For a boost along an arbitrary direction

\[ I^\mu_\nu = \begin{bmatrix} 0 & -\frac{P_x}{|p|} & -\frac{P_y}{|p|} & -\frac{P_z}{|p|} \\ -\frac{P_x}{|p|} & 0 & 0 & 0 \\ -\frac{P_y}{|p|} & 0 & 0 & 0 \\ -\frac{P_z}{|p|} & 0 & 0 & 0 \end{bmatrix} \] (80)

and

\[ S = e^{-\frac{\omega}{2} \alpha \cdot \vec{p}} \] (81)

\[ u^\nu(p) = \sqrt{\frac{m + E}{2m}} \begin{bmatrix} 1 & 0 & \frac{P_z}{E + m} & \frac{P_x - iP_y}{E + m} \\ 0 & 1 & \frac{P_x + iP_y}{E + m} & -\frac{P_z}{E + m} \\ \frac{P_z}{E + m} & \frac{P_x - iP_y}{E + m} & 1 & 0 \\ \frac{P_x + iP_y}{E + m} & -\frac{P_z}{E + m} & 0 & 1 \end{bmatrix} u^\nu(0) \] (82)

The four solutions of the spinors are
\[ u^1(p) = \sqrt{\frac{m + E}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{P_z}{E + m} \\ \frac{P_x + iP_y}{E + m} \end{bmatrix}; \quad u^2(p) = \sqrt{\frac{m + E}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{P_x - iP_y}{E + m} \\ \frac{P_z}{E + m} \end{bmatrix} \]

\[ u^3(p) = \sqrt{\frac{m + E}{2m}} \begin{bmatrix} \frac{P_z}{E + m} \\ \frac{P_x + iP_y}{E + m} \\ 1 \\ 0 \end{bmatrix}; \quad u^4(p) = \sqrt{\frac{m + E}{2m}} \begin{bmatrix} \frac{P_x - iP_y}{E + m} \\ \frac{P_z}{E + m} \\ 0 \\ 1 \end{bmatrix} \]

Equation 83

Note that Equation 83 is obtained by using a normalization:

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \]

Equation 84

We can also solve the Dirac equation directly:

\[ Eu = \left( \bar{\alpha} \cdot \vec{P} + \beta m \right) u = \begin{pmatrix} m & \bar{\sigma} \cdot \vec{P} \\ \bar{\sigma} \cdot \vec{P} & -m \end{pmatrix} u \]

Equation 85

\[ u_a \begin{pmatrix} u_A \\ u_B \end{pmatrix}, \text{ where } u_A, u_B \text{ are two-component wave functions} \]

\[ E \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} m & \bar{\sigma} \cdot \vec{P} \\ \bar{\sigma} \cdot \vec{P} & -m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} \]
We obtain the coupled equations

\[
\tilde{\sigma} \cdot \tilde{P} u_B = (E - m) u_A \\
\text{and} \\
\tilde{\sigma} \cdot \tilde{P} u_A = (E + m) u_B
\]  \hspace{1cm} (86)

A possible solution to Equation 86 is

For \( E > 0 \), \( u^{(s)} = N \left( \begin{array}{c} x^{(s)} \\ \frac{\tilde{\sigma} \cdot \tilde{P}}{E + m} x^{(s)} \end{array} \right) \), \( x^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) \( x^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) 

\( S = 1, 2 \)

For \( E < 0 \), \( u^{(s+2)} = N \left( \begin{array}{c} \frac{\tilde{\sigma} \cdot \tilde{P}}{E - m} x^{(s)} \end{array} \right) \)

Explicitly:

\[
\begin{align*}
\begin{pmatrix} 1 \\ 0 \\ Pz \\ Px + iP_y \\ E + m \\ E + m \\ E + m \end{pmatrix} & ; \quad u^1 (p) = N \left( \begin{array}{c} 0 \\ 1 \\ Px - iP_y \\ E + m \\ Pz \\ -Pz \\ E + m \end{array} \right) ; \quad E > 0 \\
\begin{pmatrix} Pz \\ E - m \\ Px + iP_y \\ E - m \\ 1 \\ 0 \\ E - m \end{pmatrix} & ; \quad u^3 (p) = N \left( \begin{array}{c} Px - iP_y \\ E - m \\ E - m \\ 0 \\ 0 \\ 1 \\ E - m \end{array} \right) ; \quad E < 0
\end{align*}
\]  \hspace{1cm} (87)
Note that \( u^3(p), u^4(p) \) from Equation 87 are different from those obtained by the boost method (Equation 83). Apart from a difference in the normalization factor \( N \),

\[
u^{(3,4)}_{\text{boost}}(p) = u^{(3,4)}_{\text{direct}}(-p), \text{ where } p = (E, \vec{p})
\]

Note that \( E > 0 \) for \( u^{(3,4)}_{\text{boost}}(p) \), while \( E < 0 \) for \( u^{(3,4)}_{\text{direct}}(p) \).

In order to interpret the \( E < 0 \) solutions for \( u^{(3,4)}(p) \), we examine the charge-conjugation transformation on the Dirac equation.

The Dirac equation for an electron in an EM field can be obtained with the gauge substitution

\[
\partial^\mu \rightarrow D^\mu \equiv \partial^\mu + iqA^\mu
\]

(89)

For an electron with charge \( q = -e \) \((e > 0)\), we have

\[
i\partial^\mu \rightarrow i\partial^\mu + eA^\mu
\]

(90)

The Dirac equation for an electron becomes

\[
\left[ \gamma^\mu \left( i\partial^\mu + eA^\mu \right) - m \right] \psi = 0
\]

(91)

The Dirac equation for a positron, \( \psi_c \), would be

\[
\left[ \gamma^\mu \left( i\partial^\mu - eA^\mu \right) - m \right] \psi_c = 0
\]

(92)

Our task is to find the transformation linking \( \psi \) to \( \psi_c \). To change the relative sign between the \( \partial^\mu \) and \( A^\mu \) terms in Equation 91, one can take a complex conjugate of Equation 91

\[
\left[ \left( \gamma^\mu \right)^* \left( -i\partial^\mu + eA^\mu \right) - m \right] \psi^* = 0
\]

(93)

To find a \( 4 \times 4 \) matrix \( S_c \), which satisfies
\[ S_c \psi^* = \psi_c \]  

(94)

We insert \( S_c \) term in front of \( \psi^* \) in Equation 93 and multiply \( S_c \) from the left:

\[
\left[ S_c \left( \gamma^\mu \right)^* S_c^{-1} \left( -i \partial_n + eA_\mu \right) - m \right] \left( S_c \psi^* \right) = 0
\]

(95)

From Equations 94 and 95, it follows that \( S_c \) satisfies the following equation:

\[ S_c \left( \gamma^\mu \right)^* S_c^{-1} = -\gamma^\mu \]

(96)

Now, the Pauli-Dirac representation of \( \gamma^\mu \) gives

\[ \left( \gamma^\mu \right)^* = \gamma^\mu \text{ for } \mu = 0, 1, 3 \]
\[ \left( \gamma^\mu \right)^* = -\gamma^\mu \text{ for } \mu = 2 \]

(97)

Therefore, Equation 96 implies that \( S_c \) commutes with \( \gamma^2 \) and anticommutes with \( \gamma^0, \gamma^1, \gamma^3 \). This can be satisfied if \( S_c \) is proportional to \( \gamma^2 \):

\[ S_c = i\gamma^2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \]

(98)

The factor \( i \) is chosen by convention.

To obtain the positron wave function, one takes a complex conjugate of the electron wave function, followed by a multiplication of \( i\gamma^2 \):

\[ \psi_+ = i\gamma^2 \psi^* \]

(99)

We can apply Equation 99 to \( \psi^{(1)}(p) \):

...
\[
\psi^{(1)}(p) = N \begin{pmatrix} 1 \\ 0 \\ \frac{P_x + iP_y}{E + m} \end{pmatrix} e^{-iP_x^2}, \quad P^0 = E > 0
\] (100)

\[
\gamma^2 \psi^{(1)*}(p) = \begin{pmatrix} 1 & 0 & \frac{P_z}{E + m} \\ P_x - iP_y & E + m & \frac{P_x - iP_y}{E + m} \end{pmatrix} N \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{iP_x^2}
\] (101)

\[
= N \begin{pmatrix} \frac{P_x - iP_y}{E + m} \\ \frac{-P_z}{E + m} \\ 0 \\ 1 \end{pmatrix} e^{iP_x^2}; \quad P^0 = E > 0
\]

We can also write \(\psi^{(4)}(p')\) as

\[
\psi^{(4)}(p') = N \begin{pmatrix} P_x' - iP_y' \\ E - m \\ \frac{-P_z'}{E - m} \\ 0 \\ 1 \end{pmatrix} e^{-iP_x^2} \quad \text{where} \quad (P')^0 = E' < 0
\] (102)

Substituting \(p' = -p\) into Equation 102, we obtain
\[
\psi^{(4)}(-p) = N \begin{pmatrix}
-Px - iPy \\
E - m \\
Pz \\
-E - m \\
0 \\
1
\end{pmatrix} e^{ip\cdot x}, \quad P^0 = E > 0
\]  

Equation 103 is identical to Equation 101.

Hence, \( \psi^{(1)}_c = i\gamma^2 \psi^{(1)*}(p) = \psi^{(4)}(-p) \)  

Similarly, \( \psi^{(2)}_c = i\gamma^2 \psi^{(2)*}(p) = -\psi^{(3)}(-p) \)  

It also follows

\[
\begin{align*}
u^{(4)}(-p) &= \nu^{(1)}(p) \\
-u^{(3)}(-p) &= \nu^{(2)}(p)
\end{align*}
\]  

where \( \nu^{(1)}, \nu^{(2)} \) are the spinors for positron wave functions.