1. [5] Bell’s inequalities from “instruction lists”. {This is a bit long, but very easy way to get Bell’s inequalities.} In class we derived/will derive one version of Bell’s inequalities, using general notions of “elements of reality”, and “locality”. Here we want to use a more concrete concept of “instruction lists”. In particular, for our local hidden variable model, we will pretend that every photon that leaves the source carries with it some sort of instruction list, as to exactly how it should behave when it meets a polarizing beam splitter at a given orientation $\theta$. For example, the list might look like: $[\theta = 0^\circ$: transmitted; $\theta = 1^\circ$: transmitted; $\theta = 2^\circ$: reflected; etc.]. For simplicity for the moment we consider a lossless system; later on we could add additional instructions like “be absorbed”, or “don’t cause the detector to fire”, to take into account losses and detector inefficiencies.

The experimental system we have in mind to model is indicated above. We have a source that emits pairs of photons, and each one going to the left (right) is analyzed in either the $\alpha_1$ or $\alpha_2$ ($\beta_1$ or $\beta_2$) basis. Thus, each pair of particles can be characterized by a four-instruction list, which completely determines what each particle would do for the two settings of its analyzer, i.e.,
1. if $\alpha = \alpha_1$, then particle 1 take the $?_1$-channel.
2. if $\alpha = \alpha_2$, then particle 1 take the $?_2$-channel.
3. if $\beta = \beta_1$, then particle 2 take the $?_3$-channel.
4. if $\beta = \beta_2$, then particle 2 take the $?_4$-channel.
(here $?_i$ is either the reflected [r] or the transmitted [t] channel of the polarizing beamsplitter).

We can use the shorthand notation $?_1?_2?_3?_4$ (e.g., “rrtr”, “rttr”, etc.) to represent a given list. There are a total of 16 such lists, but each pair of particles has only one of these 16. Note that this model is clearly “local”, since the outcome of particle 1 depends only on $\alpha$, and the outcome of particle 2 depends only on $\beta$. And it is clearly “realistic”, in the sense that the outcome of each measurement is an “element of reality”, whether or not we actually measure it. Next consider a large number of pairs, each with its own instruction list. Denote by $n_a(\alpha, \beta)$ the number of pairs with the instruction set “?t?t?” (i.e., those pairs whose first and third instructions are transmission through the PBS; the 2nd and fourth instructions can be anything). Similarly, $n_a(\alpha, \beta)$ is the number with the instruction set “r??t”, and so on.
a). Finally, realizing that every term can be expanded, for example
\[ n_{ttt}(\alpha, \beta) = n_{ttt} + n_{trt} + n_{rtt} + n_{trtr}, \]
prove that
\[ n_{tt}(\alpha_2, \beta_2) + n_{tr}(\alpha_1, \beta_2) + n_{rt}(\alpha_2, \beta_1) - n_{tt}(\alpha_1, \beta_1) \geq 0 \]  \[3\]
b). Now, let’s say we only wanted to have to look at the transmitted photons. We can convert the above inequality, by using the locality assumption to write the relation:
\[ n_{t}(\alpha_1) = n_{t}(\alpha_1, \beta_2) \]  \[= n_{tt}(\alpha_1, \beta_2) + n_{tr}(\alpha_1, \beta_2)\], and a similar relation for \( n_{t}(\beta_1) = n_{t}(\alpha_2, \beta_1) \).

These last two, by the way, are simply the number of singles counts at detector 1 when \( \alpha = \alpha_1 \) and at detector 2 when \( \beta = \beta_1 \). \[2\]

It turns out that, even if we had introduced a third sort of instruction “u” = “undetected”, the inequality you just derived in b) still holds (you can go through the algebra; it is no more difficult, but a good deal more tedious, since now there are \( 3^4 \) different lists). The point is that we now have an equality that depends only on the actual number of detected counts, and not on probabilities. Therefore, we do not need to concern ourselves with pairs we do not detect. This inequality was first derived by Clauser and Horne \[ J. F. Clauser and M. A. Horne, Phys. Rev. D 10, 526 (1974), \] and is really the “best” one to use for a completely uncontroversial test of Bell’s inequalities. As we shall see, it has never been violated in any experiment to date, contrary to popular misconception.

2. \[5\] In class, we discussed one Bell’s inequality, which constrained the value of \( |S| \leq 2 \), where \( S \) was a combination of correlation coefficient. Here we will work with a slightly different version of Bell’s inequality:
\[ P_{1,2}(\alpha, \beta) + P_{1,2}(\alpha, \beta') + P_{1,2}(\alpha', \beta) - P_{1,2}(\alpha', \beta') \leq P_1(\alpha) + P_2(\beta). \]
a) The Bell state \( |\phi^+\rangle = (|HH\rangle + |VV\rangle)/\sqrt{2} \) violates this, for instance, when \( \alpha = 0, \alpha' = 45^\circ, \beta = 22.5^\circ, \beta' = -22.5^\circ \). Prove this. \[1\]
b) Next, put in finite efficiencies \( \eta \) for each of the detectors. Assuming everything else is perfect, what is the constraint on \( \eta \) such that we can still get a violation? \[2\]
c) Above, we assumed that \( P(\alpha, \beta) = \cos^2(\beta - \alpha) \), i.e., visibility = 1. In a real experiment, however, this is never the case. So, instead write
\[ P(\alpha, \beta) = \frac{1 + V \cos \{2(\beta - \alpha)\}}{4} \]
and find the limit on \( V \) (which I hope you recognize as the fringe visibility) such that we can get a violation. (Note: The maximum value of \( V \) producable in this sort of experiment by any classical light field [i.e., describable by Maxwell’s equations] is 0.5) \[2\]

3. \[10\] Let’s say that instead of \( (|HH\rangle + |VV\rangle)/\sqrt{2} \), we consider a more general state \( (|HH\rangle + \varepsilon |VV\rangle)/\sqrt{1 + \varepsilon^2} \).
a) Calculate \( P_1(\alpha), P_2(\beta), \) and \( P_{1,2}(\alpha, \beta) \), for a given \( \varepsilon \) and detector efficiency \( \eta \). \[2\]

Next we want to find values of \( \alpha, \alpha', \beta, \beta' \) that maximize the violation (i.e., maximize
B= P_{1,2}(\alpha, \beta) + P_{1,2}(\alpha', \beta') + P_{1,2}(\alpha, \beta') - P_{1,2}(\alpha', \beta') - P_1(\alpha) - P_2(\beta). This is actually a nontrivial calculation, so we will use a few hints, which will make it easier to get to the result, which is to see how using such a non-maximally entangled state actually reduces the required detector efficiency.

Hint #1. By symmetry, it can be shown that the angles can always be chosen symmetrically, as follows (note -- we could have done this in problem 2a, if we wanted to):

Hint #2 (a biggie). It turns out that for small \( \epsilon \), our best bet is to choose \( \alpha, \beta \) to basically remove the singles counts as much as possible. Therefore, you may assume \( \alpha = \beta = 90^\circ \).

b). Write the condition for \( \eta \) (depending on \( \epsilon \) and \( \alpha' \)) such that we are at the threshold of violating Bell’s inequality, i.e., such that \( B=0 \). [3]

c). Bearing in mind that \( 0 < \eta \leq 1 \), find the minimum value of \( \eta \), and find the value of \( \alpha' \) for a given \( \epsilon \).

You may/should assume that \( \epsilon \) is \( <<1 \). [4]

Hint: Write \( \alpha' = \pi/2 + \delta \), and consider small \( \delta \).

In any real system there will also be noise (e.g., background light, detector “dark counts”, etc.). We won’t redo the general calculation with noise (that’s even more nontrivial).

d). However, state what happens to the required \( \eta \) if there is an unpolarized background in the system. [1]

4. [7] Hardy’s version of nonlocality. Imagine that we have a system with identical pairs of particles, one member of each pair going to Alice, the other member to Bob. The particles are in a state of the form: \( |\psi\rangle = a_{aa} |\alpha\alpha\rangle + a_{\alpha\alpha} |\alpha\alpha\rangle + a_{aa} |\alpha\alpha\rangle + a_{\alpha\alpha} |\alpha\alpha\rangle \) (This is a completely general state; however, for simplicity you may assume all the coefficients are real.) Alice and Bob will each randomly choose to measure in the \( \alpha, \alpha' \) or the \( \delta, \delta' \) basis. (Note: this is a bit different than the cases considered above, where Alice measured in one of the two bases \( \alpha, \alpha' \) or \( \beta, \beta' \); and Bob measured in one of these two bases \( \beta, \beta' \) or \( \beta', \beta'' \); hence there were four total bases. Here Alice will choose between the \( \alpha \) or the \( \delta \) basis (\( |\delta\rangle = (\cos \theta |\alpha\rangle + \sin \theta |\alpha'\rangle \)), and Bob will also choose between these same bases.)

Now I list the three constraints on the observations that they make:

\[ P(\alpha, \alpha) = 0, \quad P(\delta, \alpha') = P(\alpha', \delta) = 0. \]

a) What are the resulting constraints on the 4 coefficients \( a_{ij} \), in terms of \( \delta \), given the above three conditions? [3]

b) According to quantum mechanics, what is the maximum of \( P(\delta, \delta) \), and what are \( \theta \) and \( \alpha \) that give this maximum? [4]

c) Optional: Now let’s write this in terms of a state that might look more familiar: If \( |\psi\rangle = (|HH\rangle + |evV\rangle) / \sqrt{1 + \epsilon^2} \), what \( \alpha, \delta, \) and \( \epsilon \) give the above results; give your answer in terms of the “measurement angles” \( \alpha \) and \( \delta \), relative to \( H \), i.e.,

\( (|\alpha\rangle = (\cos \alpha |H\rangle + \sin \alpha |V\rangle); |\delta\rangle = (\cos \delta |H\rangle + \sin \delta |V\rangle) \). In other words, if someone gives you
the state $|\psi\rangle$, how should you measure it to satisfy the three constraints, and yet maximize $P(\delta, \delta)$? 

5. [10] The GHZ experiment (a bit simplified). The three-qubit generalization of a Bell state was first discussed by Greenberger-Horne-Zeilinger, and is therefore often called a GHZ-states. One example, written in the circular basis is: $(|L_1 L_2 L_3\rangle - |R_1 R_2 R_3\rangle)/\sqrt{2}$.

Imagine that we send each of the three particles to separated experimenters Alice, Bob, and Charley. Each of them will measure in either the H/V basis, or in the 45/-45° basis (using a polarization rotator and a polarizing beam splitter); in the diagram the H or 45° photons go to the transmitted [“t”] detectors, while the V or -45° photons go to the reflected [“r”] detectors.

a) Calculate $P(-45°, -45°, -45°)$ 

b) Calculate $P(V, -45°, -45°)$ {i.e., the probability that Alice measures Vertical polarization, and Bob and Charley both see -45°}. 

c) Now for some local realistic arguing [but still believing the results of a) and b)].... Assume that we had one of the triplets which would contribute an event to the measurement in a). *Conditional on this set*, what must be the outcome for Alice (i.e., H or V) if she were instead to have measured in the H/V basis? (Hint: Bob’s and Charley’s photons must not change their behavior if Alice changes her measurement basis -- that’s what we mean by “locality”). 

d-e) Repeat b) and c) for $P(-45°, V, -45°$) [and looking at Bob’s outcome]. 

f-g) Repeat b) and c) for $P(-45°, -45°, V)$ [and looking at Charley’s outcome].

h) From your results from c), e), and g), again conditional on a triplet which would have given a $(-45°, -45°, -45°)$ result, what must be the result if Alice, Bob, and Charley each decided they would measure in the H/V basis instead? 

i) Finally, calculate $P(HHH)$ directly. QED!