Physics 513-QOI: Quantum Optics and Quantum Information

Problem set #4 [44]
{Distributed 2/9/2016; due 2/16/2016}

Reading: N&C 2.4-2.5

1. [8] Basic spin-field interactions: Basic NMR. Consider a spin-1/2 particle with magnetic moment $\mu$, initially in the state $\frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$ (with respect to $z$), in a strong magnetic field $\vec{B} = B_0 \hat{z}$.

a) Calculate the precession rate of the spin by calculating $\langle \sigma_x(t) \rangle$, where $\sigma_x$ is the x-Pauli operator. Does the rate depend on the coefficients of the spin-up/down terms? [2]

Now apply a resonant rotating transverse magnetic field $\vec{B} = B\cos(\omega t) \hat{x} - B\sin(\omega t) \hat{y}$, where $B$ is small compared to $B_0$, and $\omega$ matches the precession rate calculated in a).

b) In the limit where we can ignore the high-frequency non-resonant terms, and assuming the spin now starts in the state $|\uparrow\rangle$ at $t = 0$, calculate the time-evolution of the state, i.e., what are the coefficients $\alpha(t)$ and $\beta(t)$ in $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$? (Note: This problem is worked out in various ways in many many textbooks. Feel free to follow your favorite…) [5]

c) Calculate the time needed to apply the oscillating field to produce the state (aside from a global phase factor) $\frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$ [1].

2. [7] Entangled states. Soon enough we will start discussing the production and uses of entangled states in detail. For 1 qubit, there are of course no entangled states. For 2 qubits, there are four such states which span the space. They are known as the Bell states. We shall look at the polarization implementation of these:

“Bell states” $|\psi^\pm\rangle = (|H\rangle|V\rangle \pm |V\rangle|H\rangle)/\sqrt{2}$

$|\phi^\pm\rangle = (|H\rangle|H\rangle \pm |V\rangle|V\rangle)/\sqrt{2}$

a) Write these in the $D, A$ basis, where

$|D\rangle \equiv 45^\circ \equiv (|H\rangle + |V\rangle)/\sqrt{2}$, $|A\rangle \equiv -45^\circ \equiv (|H\rangle - |V\rangle)/\sqrt{2}$. [2]

b) Write these in the $L, R$ basis, where

$|L\rangle \equiv (|H\rangle + i |V\rangle)/\sqrt{2}$, $|R\rangle \equiv (|H\rangle - i |V\rangle)/\sqrt{2}$. [2]

c) Finally, calculate $P(\theta_1, \theta_2) \equiv |\langle \theta_1, \theta_2 |\psi^\pm\rangle|^2$ and $|\langle \theta_1, \theta_2 |\phi^\pm\rangle|^2$. [3]
3. [6] If we have a pure state, we can always describe it simply by using a ket, i.e., \( |\psi\rangle \).
   However, we could also use the density matrix \( \rho = |\psi\rangle \langle \psi | \).
   a) Write \( \rho \) for the pure states \( |\psi\rangle = |H\rangle, |45\rangle, |-45\rangle, |L\rangle, |\theta\rangle \), and the most general state \( |\chi\rangle \), in the H/V basis, both in Dirac notation and in matrix form. [3]
   
   Ex. \( \rho = |H\rangle \langle H| \) and \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).
   
   b) Write \( \rho \) for \( |H\rangle \) and \( |45\rangle \) in the 45/-45 basis. [1]
   
   c) From our earlier problem sets, we know that the probability to get through a polarizer at angle \( \theta \) is described by \( P(\theta) = |\langle \theta | \psi \rangle|^2 \). For density matrices this is generalized to: \( \text{Tr} \ (A \rho) = \langle \theta | \rho | \theta \rangle \), where \( A \) is the projection operator for the polarizer \( |\theta\rangle \langle \theta| \). Using this formalism, calculate \( P(\theta) \) for the first 5 states above, i.e., all but the general state \( |\chi\rangle \) (yes, I know you already know the answers, and it would be easier the other way). [2]

4. [4] Now we look at the case of impure, or “mixed” states. In this case, the density matrix is a sum of density matrices for pure states: \( \rho = \sum_i w_i |\psi_i\rangle \langle \psi_i| \). (For all the problems below, please use the HV basis.)
   a) Write \( \rho \) for light that is half \( H \) and half \( V \) (incoherent sum – this means there is no definite phase relationship between \( H \) and \( V \), so there will be no cross terms like \( |H\rangle \langle V| \)). [1/2]
   b) Write \( \rho \) for light that is half \( 45^\circ \) and half \(-45^\circ \). [1/2]
   c) Write \( \rho \) for light that is 2 parts \( H \) and 1 part \( V \). [1/2]
   d) Now decompose this \( \rho \) into a sum of (completely pure density matrix) + (completely mixed density matrix). [a completely mixed density matrix has no off-diagonal elements, and equal diagonal elements] [1]
   e) Calculate \( P(\theta) \) for these, and discuss, if you feel so inclined. [1.5]

5. [5] a) Calculate \( \text{Tr} \ \rho \) and \( \text{Tr} \ \rho^2 \) for the \( \rho \)’s corresponding to \( |H\rangle, |L\rangle, \) and \( |\theta\rangle \), and for the three \( \rho \)’s from the previous problem (i.e., for a total of six \( \rho \)’s) [3]
   b) What is the significance of \( \text{Tr} \ \rho \) and \( \text{Tr} \ \rho^2 \)? [2]

6. [7 =3+1+1+2] N&C: Exercise 2.72

7. [7] Write the density matrix \( \rho \), in ket and matrix form, (in the \( |HH\rangle, |HV\rangle, |VH\rangle, |VV\rangle \) basis) for the following two-qubit states:
   a) \( |H\rangle_i |H\rangle_2 \) [1/2]
   b) \( |45\rangle_i |45\rangle_2 \) [1/2]
   c) The four Bell’s states \( \psi^+, \psi^- \) [2]
   d) The non-maximally entangled state \( |H\rangle_i |H\rangle_2 + \varepsilon |V\rangle_i |V\rangle_2 \) (the term “non-maximal” describes the situation when \( \varepsilon \neq 1 \); of course, you need to properly normalize the state.) [1]
   e) A classical mixture of 50% (both horizontal), 50% (both vertical) [1]
   f) Same as e), but 50% (both \( 45^\circ \)), or 50% (both \( -45^\circ \)) [1]
   g) A completely mixed state. [1]