Phys 509 - Problem Set 3

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1 Infinitesimal Homotopy

The infinitesimal homotopy relation reads:

$$\mathcal{L}_X\omega = \mathrm{d}\left(i_X\omega\right) + i_X\left(\mathrm{d}\omega\right)$$

Using this, we wish to evaluate $\mathcal{L}_X\left(\mathrm{d}\omega\right)$ and $\mathrm{d}\left(\mathcal{L}_X\omega\right)$. First:

$$\mathcal{L}_X\mathrm{d}\omega = \mathrm{d}\left(i_X\mathrm{d}\omega\right) + i_X\mathrm{d}\mathrm{d}\omega$$

However, since $\mathrm{d}\mathrm{d}\omega = \mathrm{d}^2\omega = 0$, this leaves us with:

$$\mathcal{L}_X\mathrm{d}\omega = \mathrm{d}\left(i_X\mathrm{d}\omega\right)$$

On the other hand, we have:

$$\begin{aligned}\mathrm{d}\left(\mathcal{L}_X\omega\right) &= \mathrm{d}\left(\mathrm{d}\left(i_X\omega\right) + i_X\mathrm{d}\omega\right)\\ &= \mathrm{d}^2\left(i_X\omega\right) + \mathrm{d}\left(i_X\mathrm{d}\omega\right)\end{aligned}$$

Once again, $\mathrm{d}^2\left(i_X\omega\right) = 0$; this leave us with:

$$\mathrm{d}\left(\mathcal{L}_X\omega\right) = \mathrm{d}\left(i_X\mathrm{d}\omega\right)$$

Comparing the two yields the desired equality:

$$\mathrm{d}\left(\mathcal{L}_X\omega\right) = \mathcal{L}_X\left(\mathrm{d}\omega\right)$$

2 Magnetic Solid

Consider a Hamiltonian:

$$H\left(x,k\right) = \mathcal{E}\left(k\right) + V\left(x\right)$$

describing an electron in a magnetic solid, where x is the position, k is the Bloch momentum, $\mathcal{E}\left(k\right)$ is the band energy, and $B\left(x\right)$ is the external magentic field.

Consider also a symplectic form:

$$\omega = \delta_{ij}\mathrm{d}k^i\wedge\mathrm{d}x^j - \frac{e}{2}\varepsilon_{ijk}B_i\left(x\right)\mathrm{d}x^j\wedge\mathrm{d}x^k + \frac{1}{2}\varepsilon_{ijk}\Omega_i\left(k\right)\mathrm{d}k^j\wedge\mathrm{d}k^k$$

Part (a)

We wish to show that the Hamiltonian vector field is:

$$X = \dot{x}^i\frac{\partial}{\partial x^i} + \dot{k}^j\frac{\partial}{\partial k^j}$$

and we wish to recover the equations of motion from:

$$\mathrm{d}H = -i_X\omega$$

Consider then that:

$$i_X\omega = \omega\left(X,\cdot\right)$$

The first term of ω yields:

$$\begin{aligned}\delta_{ij}\mathrm{d}k^i\wedge\mathrm{d}x^j\left(X,\cdot\right) &= -\delta_{ij}\dot{x}^j\mathrm{d}k^i + \delta_{ij}\dot{k}^i\mathrm{d}x^j\\ &= -\dot{x}_i\mathrm{d}k^i + \dot{k}_i\mathrm{d}x^i\end{aligned}$$

The second term of ω yields:

$$\begin{aligned}-\frac{e}{2}\varepsilon_{ijk}B_i\left(x\right)\mathrm{d}x^j\wedge\mathrm{d}x^k\left(X,\cdot\right) &= -\frac{e}{2}\varepsilon_{ijk}B_i\left(x\right)\left(\dot{x}^j\mathrm{d}x^k - \dot{x}^k\mathrm{d}x^j\right)\\ &= -e\varepsilon_{ijk}B_i\left(x\right)\dot{x}^j\mathrm{d}x^k\end{aligned}$$

The third term of ω yields:

$$\begin{aligned}\frac{1}{2}\varepsilon_{ijk}\Omega\left(k\right)\mathrm{d}k^j\wedge\mathrm{d}k^k\left(X,\cdot\right) &= \frac{1}{2}\varepsilon_{ijk}\Omega_i\left(k\right)\left(\dot{k}^j\mathrm{d}k^k - \dot{k}^k\mathrm{d}k^j\right)\\ &= \varepsilon_{ijk}\Omega_i\left(k\right)\dot{k}^j\mathrm{d}k^k\end{aligned}$$

This gives:

$$i_X\omega = \left(\dot{k}_k - \varepsilon_{ijk}B_i\left(x\right)\dot{x}^j\right)\mathrm{d}x^k + \left(-\dot{x}_k + \varepsilon_{ijk}\Omega_i\left(k\right)\dot{k}^j\right)$$

Comparing this to:

$$\begin{aligned}\mathrm{d}H &= \frac{\partial H}{\partial x^i}\mathrm{d}x^i + \frac{\partial H}{\partial k^j}\mathrm{d}k^j\\ &= \frac{\partial V}{\partial x^i}\mathrm{d}x^i + \frac{\partial\mathcal{E}}{\partial k^j}\mathrm{d}k^j\end{aligned}$$

gives us that, for $-i_X\omega = \mathrm{d}H$, the following equations of motion:

$$\begin{aligned}-\dot{x}_k + \varepsilon_{ijk}\Omega_i\left(k\right)\dot{k}^j &= -\frac{\partial\mathcal{E}}{\partial k^k}\\ \dot{k}_k - \varepsilon_{ijk}B_i\left(x\right)\dot{x}^j &= -\frac{\partial V}{\partial x^k}\end{aligned}$$

or:

$$\begin{aligned}\dot{x}_k &= -\varepsilon_{kij}\dot{k}^i\Omega_j\left(k\right) + \frac{\partial\mathcal{E}}{\partial k^k}\\ \dot{k}_k &= -\varepsilon_{kij}\dot{x}^iB_j\left(x\right) - \frac{\partial V}{\partial x^k}\end{aligned}$$

Part (b)

First we wish to show that ω is closed; that is, show $\mathrm{d}\omega = 0$. Evaluating, we get:

$$\begin{aligned}\mathrm{d}\omega &= \frac{\partial}{\partial x^l}\left(\delta_{ij}\right)\mathrm{d}x^l\wedge\mathrm{d}k^i\wedge\mathrm{d}x^j + \frac{\partial}{\partial k^l}\left(\delta_{ij}\right)\mathrm{d}k^l\wedge\mathrm{d}k^i\wedge\mathrm{d}x^j\\ &\quad -\frac{\partial}{\partial x^l}\left(\frac{e}{2}\varepsilon_{ijk}B_i\left(x\right)\right)\mathrm{d}x^l\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k\end{aligned}$$

$$
\begin{aligned}
&-\frac{\partial}{\partial k^l}\left(\frac{e}{2}\varepsilon_{ijk}B_i\left(k\right)\right)\mathrm{d}k^l\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k\\
&+\frac{\partial}{\partial x^l}\left(\frac{1}{2}\varepsilon_{ijk}\Omega_i\left(k\right)\right)\mathrm{d}x^l\wedge\mathrm{d}k^j\wedge\mathrm{d}k^k\\
&+\frac{\partial}{\partial k^l}\left(\frac{1}{2}\varepsilon_{ijk}\Omega_i\left(k\right)\right)\mathrm{d}k^l\wedge\mathrm{d}k^j\wedge\mathrm{d}k^k
\end{aligned}
$$

Eliminating the derivatives that are trivially zero, we obtain:

$$
\begin{aligned}
\mathrm{d}\omega &= -\frac{e}{2}\varepsilon_{ijk}\frac{\partial}{\partial x^l}\left(B_i\left(x\right)\right)\mathrm{d}x^l\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k\\
&\quad+\frac{1}{2}\varepsilon_{ijk}\frac{\partial}{\partial k^l}\left(\Omega_i\left(k\right)\right)\mathrm{d}k^l\wedge\mathrm{d}k^j\wedge\mathrm{d}k^k
\end{aligned}
$$

Consider now that the terms $\mathrm{d}x^l\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k$ and $\mathrm{d}k^l\wedge\mathrm{d}k^j\wedge\mathrm{d}k^k$ can be rewritten as:

$$
\begin{aligned}
\mathrm{d}x^l\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k &= \varepsilon_{ljk}\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\\
\mathrm{d}k^l\wedge\mathrm{d}k^j\wedge\mathrm{d}k^k &= \varepsilon_{ljk}\mathrm{d}k^l\wedge\mathrm{d}k^j\wedge\mathrm{d}k^k
\end{aligned}
$$

which is a consequence each variable existing in 3 dimensions. As such, we have that:

$$
\begin{aligned}
\frac{e}{2}\varepsilon_{ijk}\frac{\partial B_i}{\partial x^l}\mathrm{d}x^l\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k &= \frac{e}{2}\frac{\partial B_i}{\partial x^l}\varepsilon_{ijk}\varepsilon_{ljk}\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\\
&= \frac{e}{2}\frac{\partial B_i}{\partial x^l}2\delta_{lk}\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\\
&= e\frac{\partial B_l}{\partial x^l}\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3
\end{aligned}
$$

Note however that the term $\frac{\partial B_l}{\partial x^l}$ is:

$$
\frac{\partial B_l}{\partial x^l}=\nabla_x\cdot\mathbf{B}\left(x\right)=0
$$

so we have that:

$$
\frac{e}{2}\varepsilon_{ijk}\frac{\partial B_i}{\partial x^l}\mathrm{d}x^l\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k=0
$$

Likewise, given that $\nabla_k\cdot\mathbf{\Omega}\left(k\right)=0$, we repeat the same analysis:

$$
\begin{aligned}
\frac{1}{2}\varepsilon_{ijk}\frac{\partial\Omega_i}{\partial k^l}\mathrm{d}k^l\wedge\mathrm{d}k^j\wedge\mathrm{d}k^k &= \frac{1}{2}\frac{\partial\Omega_i}{\partial k^l}\varepsilon_{ijk}\varepsilon_{ljk}\mathrm{d}k^1\wedge\mathrm{d}k^2\wedge\mathrm{d}k^3\\
&= \frac{1}{2}\frac{\partial\Omega_i}{\partial k^l}2\cdot\delta_{il}\mathrm{d}k^1\wedge\mathrm{d}k^2\wedge\mathrm{d}k^3\\
&= \frac{\partial\Omega_l}{\partial k^l}\mathrm{d}k^1\wedge\mathrm{d}k^2\wedge\mathrm{d}k^3\\
&= \nabla\cdot\Omega\mathrm{d}k^1\wedge\mathrm{d}k^2\wedge\mathrm{d}k^3\\
&= 0
\end{aligned}
$$

Hence, we have that:

$$
\mathrm{d}\omega=0
$$

Now we wish to compute the Poisson brackets $\{x,x\}$, $\{x,k\}$, and $\{k,k\}$.

In general, the Poisson bracket of two functions f and g is defined as:

$$
\{f,g\}=V_f\left(g\right)
$$

where v_f is the Hamiltonian vector field satisfying:

$$
i_{V_f}\omega=\mathrm{d}f
$$

It is clear then that we should first compute the Hamiltonian vector fields V_{x^a} corresponding to x^a, and V_{k^b} corresponding to k^b.

Poisson bracket $\left\{x^a,x^b\right\}=V_{x^a}\left(x^b\right)$.

First, to solve V_{x^a}. Let $V_{x^a}=\alpha_a^\mu\partial_{x^\mu}+\beta_a^\nu\partial_{k^\nu}$; then we have that:

$$
\begin{aligned}
i_{V_{x^a}}\omega &= \omega\left(V_{x^a},\cdot\right)\\
&= \left(\beta_a^k-e\varepsilon_{ijk}B_i\alpha_a^j\right)\mathrm{d}x^k+\left(-\alpha_a^k+\varepsilon_{ijk}\Omega_i\beta_a^j\right)\mathrm{d}k^k
\end{aligned}
$$

Solving for $-i_{V_{x^a}}\omega=\mathrm{d}x^a$ and matching the coefficients, we obtain that:

$$
\begin{aligned}
\beta_a^k-e\varepsilon_{ijk}B_i\alpha_a^j &= -\delta_{ka} &(1)\\
\varepsilon_{ijk}\Omega_i\beta_a^j-\alpha_a^k &= 0 &(2)
\end{aligned}
$$

we wish now to solve for the coefficients α and β. Solving for β in terms of α in equation 1 and then substituting into equation 2, we obtain that:

$$
\begin{aligned}
\beta_a^k &= -\delta_{ka}+e\varepsilon_{ijk}B_i\alpha_a^j\\
\alpha_a^k &= \varepsilon_{ijk}\Omega_i\beta_a^j\\
&= \varepsilon_{ijk}\Omega_i\left(-\delta_{ja}+e\varepsilon_{\mu\nu j}B_\mu\alpha_a^\nu\right)\\
&= -\varepsilon_{ijk}\Omega_i\delta_{ja}+eB_\mu\Omega_i\alpha_a^\nu\varepsilon_{ijk}\varepsilon_{\mu\nu j}
\end{aligned}
$$

Using now $\varepsilon_{ijk}\varepsilon_{\mu\nu j}=\delta_{k\mu}\delta_{i\nu}-\delta_{k\nu}\delta_{i\mu}$, we obtain that:

$$
\alpha_a^k=-\varepsilon_{iak}\Omega_i+eB_k\Omega_i\alpha_a^i-eB_i\Omega_i\alpha_a^k
$$

We can re-write this as a matrix equation:

$$
\left(\delta_{ki}\sigma-eB_k\Omega_i\right)\alpha_a^i=-\varepsilon_{sak}\Omega_s \qquad (3)
$$

where $\sigma=1+eB_i\Omega_i=1+e\mathbf{B}\cdot\mathbf{\Omega}$. Consider now that the matrix $M_{ki}=\delta_{ki}\sigma-eB_k\Omega_i$ has a particularly simple inverse; it is a matrix of the form $M=\mathbb{I}\sigma-e\mathbf{B}\mathbf{\Omega}^t$, whose inverse is $\frac{1}{\sigma}\mathbb{I}+\frac{e}{\sigma}\mathbf{\Omega}\mathbf{B}^t$ (see Sherman-Morrison-Woodbury inversion formula). In terms of coordinates, this means that:

$$
\left(M^{-1}\right)_{pk}=\frac{1}{\sigma}\delta_{pk}+\frac{e}{\sigma}\Omega_pB_k
$$

so we can explicitly solve for our coefficients α_a^p as:

$$
\begin{aligned}
\alpha_a^p &= -\left(M^{-1}\right)_{pk}\varepsilon_{sak}\Omega_s\\
&= -\frac{1}{\sigma}\delta_{pk}\varepsilon_{sak}\Omega_s-\frac{e}{\sigma}\Omega_pB_k\varepsilon_{sak}\Omega_s\\
&= -\frac{1}{\sigma}\varepsilon_{sap}\Omega_s-\frac{e}{\sigma}B_k\varepsilon_{sak}\Omega_p\Omega_s
\end{aligned}
$$

finally, noting that $\varepsilon_{sak}\Omega_p\Omega_s=\Omega\times\Omega=0$, we have that:

$$
\begin{aligned}
\alpha_a^p &= -\frac{1}{\sigma}\varepsilon_{sap}\Omega_s\\
&= -\frac{\varepsilon_{sap}\Omega_s}{1+eB_i\Omega_i}
\end{aligned}
$$

At this point, we can calculate the Poisson bracket $\left\{x^a,x^b\right\}$, because we will just have that:

$$
\begin{aligned}
\left\{x^a,x^b\right\} &= V_{x^a}\left(x^b\right)\\
&= \alpha_a^\mu\frac{\partial}{\partial x^\mu}\left(x^b\right)+\beta_\alpha^\mu\frac{\partial}{\partial k^\mu}\left(x^b\right)\\
&= \alpha_a^\mu\frac{\partial}{\partial x^\mu}\left(x^b\right)=\alpha_a^b\\
&= -\frac{\varepsilon_{sab}\Omega_s}{1+eB_i\Omega_i}=-\frac{\varepsilon_{abs}\Omega_s}{1+eB_i\Omega_i}
\end{aligned}
$$

so we get:

$$\left\{x^a, x^b\right\} = -\frac{\varepsilon_{sab}\Omega_s}{1+eB_i\Omega_i} \tag{4}$$

Poisson Bracket $\left\{x^a, k^b\right\} = V_{x^a}\left(k^b\right)$

Having now solved for α_a^p in V_{x^a}, we can proceed to solve for β_a^p. Using equation 1, we arrive at:

$$\begin{aligned}
\beta_a^p &= -\delta_{pa} + e\varepsilon_{ijp}B_i\alpha_a^j \\
&= -\delta_{pa} - \frac{e}{\sigma}B_i\Omega_s\varepsilon_{ijp}\varepsilon_{saj} \\
&= -\delta_{pa} - \frac{e}{\sigma}B_a\Omega_p + \frac{e}{\sigma}B_s\Omega_s\delta_{pa} \\
&= -\frac{1}{\sigma}\left(\delta_{pa}\sigma + eB_a\Omega_p - eB_s\Omega_s\delta_{pa}\right) \\
&= -\frac{1}{\sigma}\left(\delta_{pa} + eB_a\Omega_p\right) \\
&= -\frac{\delta_{pa} + eB_a\Omega_p}{1+eB_s\Omega_s}
\end{aligned}$$

Now, for $\left\{x^a, k^b\right\} = V_{x^a}\left(k^b\right)$, we have:

$$\begin{aligned}
\left\{x_a, k^b\right\} &= V_{x^a}\left(k^b\right) \\
&= \alpha_a^\mu\frac{\partial}{\partial x^\mu}\left(k^b\right) + \beta_a^\mu\frac{\partial}{\partial k^\mu}\left(k^b\right) \\
&= \beta_a^b = -\frac{\delta_{pa} + eB_a\Omega_p}{1+eB_s\Omega_s}
\end{aligned}$$

Hence we have:

$$\left\{x_a, k^b\right\} = -\frac{\delta_{pa} + eB_a\Omega_p}{1+eB_s\Omega_s} \tag{5}$$

Poisson Bracket $\left\{k^a, k^b\right\} = V_{k^a}\left(k^b\right)$

Consider now we wish to solve for the Hamiltonian vector field V_{k^a}. The setup is similar; as before, let $V_{k^a} = \xi_a^\mu\partial_{x^\mu} + \eta_a^\mu\partial_{k^\mu}$, and solve $i_{V_{k^a}}\omega = \mathrm{d}k^a$. This leads to the set of equations:

$$\begin{aligned}
\eta_a^k - e\varepsilon_{ijk}B_i\xi_a^j &= 0 \\
\varepsilon_{ijk}\Omega_i\eta_a^j - \xi_a^k &= -\delta_{ka}
\end{aligned}$$

Repeating the analysis gives us:

$$\eta_a^k - e\varepsilon_{iak}B_i - eB_i\Omega_k\eta_a^i + eB_i\Omega_i\eta_a^k = 0$$

or rearranging:

$$\left(\sigma\delta_{ki} - e\Omega_kB_i\right)\eta_a^i = e\varepsilon_{iak}B_i$$

Note that this equation is similar, but not the same as, equation 3. In particular, we have that the roles of $e\mathbf{B}$ and $\mathbf{\Omega}$ are interchanged. The Woodbury inversion formula yields for us then that the inverse of $M_{ki} = \sigma\delta_{ki} - e\Omega_kB_i$ is:

$$\left(M^{-1}\right)_{pk} = \frac{1}{\sigma}\delta_{pk} + \frac{e}{\sigma}\Omega_pB_k$$

so we have that:

$$\begin{aligned}
\eta_a^p &= e\left(M^{-1}\right)_{pk}\varepsilon_{sak}B_s \\
&= e\left(\frac{1}{\sigma}\delta_{pk} + \frac{e}{\sigma}\Omega_pB_k\right)\varepsilon_{sak}B_s \\
&= \frac{e}{\sigma}\varepsilon_{sap}B_s + \frac{e^2}{\sigma}\Omega_p\varepsilon_{sak}B_kB_s
\end{aligned}$$

Once again $\varepsilon_{sak}B_kB_s = 0$, so we have that:

$$\eta_a^p = \frac{e}{\sigma}\varepsilon_{sap}B_s$$

As such, we have that:

$$\begin{aligned}
\left\{k^a, k^b\right\} &= V_{k^a}\left(k^b\right) \\
&= \xi_a^\mu\partial_{x^\mu}\left(k^b\right) + \eta_a^\mu\partial_{k^\mu}\left(k^b\right) \\
&= \eta_a^b = \frac{e}{\sigma}\varepsilon_{sap}B_s
\end{aligned}$$

As such, we have that:

$$\left\{k^a, k^b\right\} = \frac{e\varepsilon_{abs}B_s}{1+e\mathbf{B}\cdot\mathbf{\Omega}}$$

Part (c)

We wish to compute the invariant volume form $\frac{1}{3!}\omega\wedge\omega\wedge\omega$. Consider first, however, that for ω is of the form:

$$\omega = \underbrace{\delta_{jk}\mathrm{d}k^j\wedge\mathrm{d}x^k}_{a} + \underbrace{P_{jk}\mathrm{d}x^j\wedge\mathrm{d}x^k}_{b} + \underbrace{Q_{jk}\mathrm{d}k^j\wedge\mathrm{d}k^k}_{c}$$

where $P_{jk} = -\frac{e}{2}\varepsilon_{ijk}B_i(x)$ and $Q_{jk} = \frac{1}{2}\varepsilon_{ijk}\Omega_i$. I have partitioned ω into terms a, b, c for later convenience.

Since each x and k run over indicies $j = 1, 2, 3$ only, then any wedge product containing more than 3 $\mathrm{d}x$'s or 3 $\mathrm{d}k$'s will vanish. As such, the only non-vanishing wedge products in $\omega\wedge\omega\wedge\omega$ are:

$$\begin{aligned}
&\delta_{j_1k_1}\delta_{j_2k_2}\delta_{j_3k_3}\cdot\mathrm{d}k^{j_1}\wedge\mathrm{d}x^{k_1}\wedge\mathrm{d}k^{j_2}\wedge\mathrm{d}x^{k_2}\wedge\mathrm{d}k^{j_3}\wedge\mathrm{d}x^{k_3} && (aaa)\\
&\delta_{j_1k_1}P_{j_2k_2}Q_{j_3k_3}\cdot\mathrm{d}k^{j_!}\wedge\mathrm{d}x^{k_1}\wedge\mathrm{d}x^{j_2}\wedge\mathrm{d}x^{k_2}\wedge\mathrm{d}k^{j_3}\wedge\mathrm{d}k^{k_3} && (abc)\\
&\delta_{j_1k_1}Q_{j_2k_2}P_{j_3k_3}\cdot\mathrm{d}k^{j_!}\wedge\mathrm{d}x^{k_1}\wedge\mathrm{d}k^{j_2}\wedge\mathrm{d}k^{k_2}\wedge\mathrm{d}x^{j_3}\wedge\mathrm{d}x^{k_3} && (acb)\\
&P_{j_1k_1}\delta_{j_2k_2}Q_{j_3k_3}\cdot\mathrm{d}x^{j_1}\wedge\mathrm{d}x^{k_!}\wedge\mathrm{d}x^{j_2}\wedge\mathrm{d}x^{k_2}\wedge\mathrm{d}k^{j_3}\wedge\mathrm{d}k^{k_3} && (bac)\\
&P_{j_1k_1}Q_{j_2k_2}\delta_{j_3k_3}\cdot\mathrm{d}x^{j_1}\wedge\mathrm{d}x^{k_!}\wedge\mathrm{d}k^{j_2}\wedge\mathrm{d}k^{k_2}\wedge\mathrm{d}k^{j_3}\wedge\mathrm{d}x^{k_3} && (bca)\\
&Q_{j_!k_1}\delta_{j_2k_2}P_{j_3k_3}\cdot\mathrm{d}k^{j_1}\wedge\mathrm{d}k^{k_1}\wedge\mathrm{d}x^{j_2}\wedge\mathrm{d}x^{k_2}\wedge\mathrm{d}x^{j_3}\wedge\mathrm{d}x^{k_3} && (cab)\\
&Q_{j_1k_1}P_{j_2k_2}\delta_{j_3k_3}\cdot\mathrm{d}k^{j_1}\wedge\mathrm{d}k^{k_1}\wedge\mathrm{d}x^{j_2}\wedge\mathrm{d}x^{k_2}\wedge\mathrm{d}k^{j_3}\wedge\mathrm{d}x^{k_3} && (cba)
\end{aligned}$$

Now, with great patience, we can simplify these terms by playing index games and permuting the basis elements.

The first term (aaa) can be brought into canonical form $\mathrm{d}k^1\wedge\mathrm{d}k^2\wedge\mathrm{d}k^3\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 = \mathrm{d}^3k\mathrm{d}^3x$ by simply permuting the basis elements, and obtaining.

$$\begin{aligned}
(aaa) &= -\delta_{j_1k_1}\delta_{j_2k_2}\delta_{j_3k_3}\cdot\mathrm{d}k^{j_1}\wedge\mathrm{d}k^{j_2}\wedge\mathrm{d}k^{j_3}\wedge\mathrm{d}x^{k_!}\wedge\mathrm{d}x^{k_2}\wedge\mathrm{d}x^{k_3} \\
&= -6\mathrm{d}k^1\wedge\mathrm{d}k^2\wedge\mathrm{d}k^3\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \\
&= -6\mathrm{d}^3k\mathrm{d}^3x
\end{aligned}$$

The other terms are actually all just index permutations of the term:

$$P_{\mu_1\mu_2}Q_{\nu_1\nu_2}\varepsilon_{s\mu_!\mu_2}\varepsilon_{s\nu_!\nu_2}\mathrm{d}^3k\mathrm{d}^3x$$

Hence we have that:

$$\begin{aligned}
(\text{everything else}) &= 6P_{\mu_1\mu_2}Q_{\nu_1\nu_2}\varepsilon_{s\mu_!\mu_2}\varepsilon_{s\nu_!\nu_2}\mathrm{d}^3k\mathrm{d}^3x \\
&= 6P_{\mu_1\mu_2}Q_{\mu_1\mu_2} - 6P_{\mu_1\mu_2}Q_{\mu_2\mu_1}\mathrm{d}^3k\mathrm{d}^3x \\
&= 12P_{\mu_1\mu_2}Q_{\mu_1\mu_2}\mathrm{d}^3k\mathrm{d}^3x
\end{aligned}$$

because Q is anti-symmetric. Finally, putting in the form of P and Q yields:

$$\begin{aligned}(\text{everything else}) &= -12 \cdot \frac{e}{4}\varepsilon_{i\mu_1\mu_2}\varepsilon_{j\mu_1\mu_2}B_i\Omega_j \mathrm{d}^3k\mathrm{d}^3x \\ &= -6e\delta_{ij}B_i\Omega_j \mathrm{d}^3k\mathrm{d}^3x \\ &= -6e\mathbf{B}\cdot\mathbf{\Omega}\mathrm{d}^3k\mathrm{d}^3x\end{aligned}$$

Hence, we have that:

$$\frac{1}{3!}\omega\wedge\omega\wedge\omega = -\left(1+e\mathbf{B}\cdot\mathbf{\Omega}\right)\mathrm{d}^3k\mathrm{d}^3x$$

3 Non-Abelian Gauge Fields

Consider a generalization of the vector pote ntial $A = A_\mu \mathrm{d}x^\mu$, where $A_\mu \in \mathrm{M}_n$ are matrix elements; that is, each A_μ carries with it matrix indicies $(A_\mu)_{ij}$.

The Maxwell tensor is then defined as:

$$F = \mathrm{d}A + A\wedge A$$

Part (i)

Show: $A\wedge A = \frac{1}{2}\left[A_\mu, A_\nu\right]\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$

In general, given a rank two tensor $B_{\mu\nu}$, we can always decompose the components $B^{\mu\nu}$ into a symmetric and anti-symmetric part as follows:

$$\begin{aligned}B_{\mu\nu} &= \left(\frac{1}{2}B_{\mu\nu}+\frac{1}{2}B_{\nu\mu}\right)+\left(\frac{1}{2}B_{\mu\nu}-\frac{1}{2}B_{\nu\mu}\right) \\ &= \mathcal{S}_{\mu\nu}+\mathcal{A}_{\nu\mu}\end{aligned}$$

In the case of a 2-form $B = B_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$, this simplification further gives us:

$$\begin{aligned}B &= B_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \\ &= \mathcal{S}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu + \mathcal{A}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \\ &= \mathcal{A}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\end{aligned}$$

where in the last equality, it was made use of the fact that:

$$\begin{aligned}\mathcal{S}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu &= \mathcal{S}_{\nu\mu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \\ &= -\mathcal{S}_{\nu\mu}\mathrm{d}x^\nu\wedge\mathrm{d}x^\mu \\ &= -\mathcal{S}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\end{aligned}$$

which can only be true if $\mathcal{S}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu = 0$. As such, we have that:

$$\begin{aligned}B &= B_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \\ &= \mathcal{A}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \\ &= \frac{1}{2}\left(B_{\mu\nu}-B_{\nu\mu}\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \qquad (6)\end{aligned}$$

Note that no assumptions as been made about the commutativity of the components of B.

In our case, let $B = A\wedge A$; this gives us $B_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu = (A_\mu\cdot A_\nu)\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$ (the dot "$\cdot$" denotes matrix multiplication of the components of A). Upon substitution into equation 6, we get:

$$\begin{aligned}A\wedge A &= \frac{1}{2}\left(A_\mu\cdot A_\nu - A_\nu\cdot A_\mu\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \\ &= \frac{1}{2}\left[A_\mu, A_\nu\right]\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \qquad (7)\end{aligned}$$

Show: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

From the definition of $F = \mathrm{d}A + A\wedge A$, we need only compute the components of $\mathrm{d}A + A\wedge A$ and compare to the components of $F = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$. Having already found that $A\wedge A = \frac{1}{2}\left[A_\mu, A_\nu\right]\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$, it only remains to calculate $\mathrm{d}A$.

For $A = A_\nu\mathrm{d}x^\nu$, we have that:

$$\mathrm{d}A = \partial_\mu A_\nu \mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$$

Using again equation 6, this time with $B_{\mu\nu} = \partial_\mu A_\nu$, we obtain:

$$\mathrm{d}A = \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$$

As such, we have that:

$$F = \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$$

from which we deduce:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Part (ii)

Define the gauge-covariant derivative to be:

$$\nabla_\mu = \partial_\mu + A_\mu$$

We wish to show that $[\nabla_\mu, \nabla_\nu] = F_{\mu\nu}$.

Consider that:

$$\begin{aligned}[\nabla_\mu, \nabla_\nu] &= [\partial_\mu + A_\mu, \partial_\nu + A_\nu] \\ &= [\partial_\mu, \partial_\nu] + [\partial_\mu, A_\nu] + [A_\mu, \partial_\nu] + [A_\mu, A_\nu]\end{aligned}$$

It is clear that $[\partial_\mu, \partial_\nu] = 0$; furthermore, $[A_\mu, A_\nu]$ is exactly the last term of $F_{\mu\nu}$. Hence, if we can show that:

$$[\partial_\mu, A_\nu] + [A_\mu, \partial_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

then we are done. To this end, for f is a differentiable function, we have:

$$\begin{aligned}[\partial_\mu, A_\nu]\circ(f) &= \partial_\mu\left(A_\nu f\right) - A_\nu\left(\partial_\mu f\right) \\ &= \left(\partial_\mu A_\nu\right) f + A_\nu\left(\partial_\mu f\right) - A_\nu\left(\partial_\mu f\right) \\ &= \left(\partial_\mu A_\nu\right) f\end{aligned}$$

so we have $[\partial_\mu, A_\nu] = \partial_\mu A_\nu$. Now, to compute $[A_\mu, \partial_\nu]$, we need only recognize that:

$$\begin{aligned}[A_\mu, \partial_\nu] &= -[\partial_\nu, A_\mu] \\ &= -\partial_\nu A_\mu\end{aligned}$$

Therefore, we have that:

$$[\partial_\mu, A_\nu] + [A_\mu, \partial_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

as desired. This shows that:

$$[\nabla_\mu, \nabla_\nu] = F_{\mu\nu}$$

Part (iii)

Given g is an invertible matrix $g \in \mathrm{GL}_n$, we wish to show that for $g \to g + \delta g$, the inverse matrix goes to $g^{-1} \to g^{-1} - g^{-1} \cdot \delta g \cdot g^{-1}$.

Consider that for $g \cdot g^{-1} = \mathbb{I}$, let $g \to g + \delta g$ and $g^{-1} \to g^{-1} + \delta\left(g^{-1}\right)$. The goal is to solve for $\delta\left(g^{-1}\right)$. Under this transformation, we must preserve:

$$(g + \delta g) \cdot \left(g^{-1} + \delta\left(g^{-1}\right)\right) = \mathbb{I}$$

This gives us:

$$\begin{aligned} \mathbb{I} &= (g + \delta g) \cdot \left(g^{-1} + \delta\left(g^{-1}\right)\right) \\ &= g \cdot g^{-1} + \delta g \cdot g^{-1} + g \cdot \delta\left(g^{-1}\right) + O\left(\delta^2\right) \end{aligned}$$

For $g \cdot g^{-1} = \mathbb{I}$, this gives us:

$$0 = \delta g \cdot g^{-1} + g \cdot \delta\left(g^{-1}\right)$$

Multiplication on the left by g^{-1} gives us then:

$$0 = g^{-1} \cdot \delta g \cdot g^{-1} + \delta\left(g^{-1}\right)$$

or:

$$\delta\left(g^{-1}\right) = -g^{-1} \cdot \delta g \cdot g^{-1}$$

Part (iv)

We wish to show that a necesssary condition for A to be a "pure gauge" is that $F = 0$. In other words, we wish to show that if $\exists g$ s.t. $A = g^{-1}\mathrm{d}g$, then $F = 0$.

First, we must compute what the components of A are in the basis $\mathrm{d}x$. For g is a (matrix-valued) rank-0 tensor, we have that:

$$\mathrm{d}g = \partial_\nu g \cdot \mathrm{d}x^\nu$$

As such, we have $A = g^{-1}\mathrm{d}g = g^{-1} \cdot \partial_\nu g \cdot \mathrm{d}x^\nu$, or:

$$A_\nu = g^{-1} \cdot \partial_\nu g$$

Now, for $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, if we can show that $\partial_\mu A_\nu - \partial_\nu A_\mu = -[A_\mu, A_\nu]$, then we are done.

Consider first that:

$$\begin{aligned} [A_\mu, A_\nu] &= A_\mu \cdot A_\nu - A_\nu \cdot A_\mu \\ &= g^{-1} \cdot \partial_\mu g \cdot g^{-1} \cdot \partial_\nu g - g^{-1} \cdot \partial_\nu g \cdot g^{-1} \cdot \partial_\mu g \quad (8) \end{aligned}$$

Now compare to:

$$\begin{aligned} \partial_\mu A_\nu - \partial_\nu A_\mu &= \partial_\mu\left(g^{-1}\right) \cdot \partial_\nu g + g^{-1} \cdot \partial_\mu \partial_\nu g \\ &\quad -\partial_\nu\left(g^{-1}\right) \cdot \partial_\mu g - g^{-1} \cdot \partial_\nu \partial_\mu g \\ &= \partial_\mu\left(g^{-1}\right) \cdot \partial_\nu g - \partial_\nu\left(g^{-1}\right) \cdot \partial_\mu g \quad (9) \end{aligned}$$

Note finally that $\partial_\mu\left(g^{-1}\right) = -g^{-1} \cdot \partial_\mu g \cdot g^{-1}$. That this is true can be seen as follows: for $g \cdot g^{-1} = \mathbb{I}$, then we have:

$$\partial_\mu\left(g \cdot g^{-1}\right) = \partial_\mu(\mathbb{I}) = 0$$

but by Leibnitz's rule, we have:

$$\partial_\mu\left(g \cdot g^{-1}\right) = \partial_\mu g \cdot g^{-1} + g \cdot \partial_\mu\left(g^{-1}\right)$$

Hence, we have:

$$\partial_\mu g \cdot g^{-1} + g \cdot \partial_\mu\left(g^{-1}\right) = 0$$

Left multiplication by g^{-1} gives:

$$\partial_\mu\left(g^{-1}\right) = -g^{-1} \cdot \partial_\mu g \cdot g^{-1} \quad (10)$$

Substitution into equation 9 gives us:

$$\partial_\mu A_\nu - \partial_\nu A_\mu = -g^{-1} \cdot \partial_\mu g \cdot g^{-1} \cdot \partial_\nu g + g^{-1} \cdot \partial_\nu \cdot g^{-1} \cdot \partial_\mu g$$

Comparing to equation 8, we find that:

$$\partial_\mu A_\nu - \partial_\nu A_\mu = -[A_\mu, A_\nu]$$

as desired. Hence, $F_{\mu\nu} = 0$.

Part (v)

Consider a gauge transform of the form:

$$\tilde{A} = g^{-1}Ag + g^{-1}\mathrm{d}g$$

which in components is:

$$\tilde{A}_\mu = g^{-1}A_\mu g + g^{-1}\left(\partial_\mu g\right)$$

we wish to show that the corresponding Maxwell tensor is:

$$\tilde{F} = g^{-1}Fg$$

Covariant Derivative Method

Under this gauge transformation, the covariant derivative transforms as:

$$\tilde{\nabla}_\mu = g^{-1}\nabla_\mu g$$

where the term $\nabla_\mu g$ means that ∇_μ is composed (as a function) with g; in other words, for f is a function we have $\left(\nabla_\mu g\right)(f) = \nabla_\mu(gf) = \nabla_\mu(g)f + g\nabla_\mu(f)$. As such, we have that:

$$\begin{aligned} \tilde{\nabla}_\mu(f) &= g^{-1}\nabla_\mu(gf) = g^{-1}\partial_\mu(gf) + g^{-1}A_\mu gf \\ &= g^{-1}g\left(\partial_\mu f\right) + g^{-1}\left(\partial_\mu g\right)f + g^{-1}A_\mu gf \\ &= \partial_\mu f + g^{-1}\partial_\mu f + g^{-1}A_\mu gf \\ &= \left(\partial_\mu + g^{-1}\partial_\mu + g^{-1}A_\mu g\right)f \\ &= \left(\partial_\mu + \tilde{A}_\mu\right)f \end{aligned}$$

so $\tilde{\nabla} = g^{-1}\nabla g$. Now, the computation of $\tilde{F}_{\mu\nu} = \left[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu\right]$ becomes simply:

$$\begin{aligned} \tilde{F}_{\mu\nu} &= \left[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu\right] = \tilde{\nabla}_\mu\tilde{\nabla}_\mu - \tilde{\nabla}_\nu\tilde{\nabla}_\mu \\ &= g^{-1}\nabla_\mu g g^{-1}\nabla_\nu g - g^{-1}\nabla_\nu g g^{-1}\nabla_\mu g \\ &= g^{-1}\nabla_\mu\nabla_\nu g - g^{-1}\nabla_\nu\nabla_\mu g \\ &= g^{-1}\left[\nabla_\mu, \nabla_\nu\right]g = g^{-1}F_{\mu\nu}g \end{aligned}$$

Direct Computation

First consider that:

$$\tilde{F} = \mathrm{d}\tilde{A} + \tilde{A} \wedge \tilde{A}$$

by definition. Computing, we have that:

$$\begin{aligned} \mathrm{d}\tilde{A} &= \mathrm{d}\left(g^{-1}Ag + g^{-1}\mathrm{d}g\right) \\ &= \mathrm{d}\left(g^{-1}Ag\right) + \mathrm{d}\left(g^{-1}\mathrm{d}g\right) \\ &= \mathrm{d}\left(g^{-1}\right) \wedge Ag + g^{-1}\mathrm{d}Ag - g^{-1}A \wedge \mathrm{d}g + \mathrm{d}\left(g^{-1}\right) \wedge \mathrm{d}g \end{aligned}$$

where in the last line, the negative sign comes from the fact that d is an anti-derivation.

Continuing, we also have that:

$$\begin{aligned} \tilde{A} \wedge \tilde{A} &= \left(g^{-1}Ag + g^{-1}\mathrm{d}g\right) \wedge \left(g^{-1}Ag + g^{-1}\mathrm{d}g\right) \\ &= g^{-1}Ag \wedge g^{-1}Ag + g^{-1}Ag \wedge g^{-1}\mathrm{d}g \\ &\quad + g^{-1}\mathrm{d}g \wedge g^{-1}Ag + g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \\ &= g^{-1}Agg^{-1} \wedge Ag + g^{-1}Agg^{-1} \wedge \mathrm{d}g \\ &\quad + g^{-1}\mathrm{d}g \cdot g^{-1} \wedge Ag + g^{-1}\mathrm{d}g \cdot g^{-1} \wedge \mathrm{d}g \end{aligned}$$

Note now that the factors $g^{-1}\mathrm{d}g \cdot g^{-1} = -\mathrm{d}\left(g^{-1}\right)$; that this is true can be directly verified from equation 10. As such, we obtain:

$$\begin{aligned} \tilde{A} \wedge \tilde{A} &= g^{-1} \cdot A \wedge A \cdot g + g^{-1} \cdot A \wedge \mathrm{d}g \\ &\quad - \mathrm{d}\left(g^{-1}\right) \wedge Ag - \mathrm{d}\left(g^{-1}\right) \wedge \mathrm{d}g \end{aligned}$$

So the sum of the two terms gives us:

$$\begin{aligned} \tilde{F} &= \mathrm{d}\tilde{A} + \tilde{A} \wedge \tilde{A} \\ &= \mathrm{d}\left(g^{-1}\right) \wedge A \cdot g + g^{-1}\mathrm{d}A \cdot g - g^{-1}A \wedge \mathrm{d}g + \mathrm{d}\left(g^{-1}\right) \wedge \mathrm{d}g \\ &\quad + g^{-1} \cdot A \wedge A \cdot g + g^{-1} \cdot A \wedge \mathrm{d}g \\ &\quad - \mathrm{d}\left(g^{-1}\right) \wedge A \cdot g - \mathrm{d}\left(g^{-1}\right) \wedge \mathrm{d}g \\ &= g^{-1}\mathrm{d}A \cdot g + g^{-1} \cdot A \wedge A \cdot g \\ &= g^{-1} \cdot (\mathrm{d}A + A \wedge A) \cdot g \\ &= g^{-1} \cdot F \cdot g \end{aligned}$$

which gives us our desired result:

$$\tilde{F} = g^{-1} \cdot F \cdot g$$

Part (vi)

We wish to show that F satisfies the Bianchi identity:

$$\mathrm{d}F - F \wedge A + A \wedge F = 0$$

For $F = \mathrm{d}A + A \wedge A$, we have that:

$$\begin{aligned} \mathrm{d}F &= \mathrm{d}\left(\mathrm{d}A + A \wedge A\right) \\ &= \mathrm{d}^2A + \mathrm{d}\left(A \wedge A\right) \\ &= \mathrm{d}\left(A \wedge A\right) \\ &= \mathrm{d}A \wedge A - A \wedge \mathrm{d}A \end{aligned}$$

where in the last equality, we have used the fact that $\mathrm{d}\left(a \wedge b\right) = \mathrm{d}a \wedge b + (-1)^p\, a \wedge \mathrm{d}b$, where p is the rank of a; in our case, A is a 1-form so $p = 1$. Continuing, we have that $\mathrm{d}A = F - A \wedge A$, so:

$$\begin{aligned} \mathrm{d}F &= (F - A \wedge A) \wedge A - A \wedge (F - A \wedge A) \\ &= F \wedge A - A \wedge F - A \wedge A \wedge A + A \wedge A \wedge A \\ &= F \wedge A - A \wedge F \end{aligned}$$

As such, we have that:

$$\mathrm{d}F - F \wedge A + A \wedge F = 0$$

Part (vii)

We wish to show that $\mathrm{d}\mathrm{Tr}\left(F \wedge F\right) = 0$, where Tr is taken over the matrix indices.

First, we show that for ω is any p-form with $n \times n$ matrix entries, we have that:

$$\mathrm{d}\left(\mathrm{Tr}\omega\right) = \mathrm{Tr}\left(\mathrm{d}\omega\right)$$

which comes from the fact that:

$$\frac{\partial}{\partial x^\mu}\left(\omega_{\nu_1 \ldots \nu_p}\right)_{ij} = \left(\frac{\partial}{\partial x^\mu}\omega_{\nu_1 \ldots \nu_p}\right)_{ij}$$

where i, j is the i, j^{th} matrix entry of $\omega_{\nu_1 \ldots \nu_p}$. Contracting $i \leftrightarrow j$, we get:

$$\frac{\partial}{\partial x^\mu}\mathrm{Tr}\left(\omega_{\nu_1 \ldots \nu_p}\right) = \mathrm{Tr}\left(\frac{\partial}{\partial x^\mu}\omega_{\nu_1 \ldots \nu_p}\right)$$

Hence, for:

$$\omega = \frac{1}{p!}\omega_{\nu_1 \ldots \nu_p}\mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_p}$$

we have that:

$$\begin{aligned} \mathrm{Tr}\left(\mathrm{d}\omega\right) &= \frac{1}{p!}\mathrm{Tr}\left(\frac{\partial}{\partial x^\mu}\omega_{\nu_1 \ldots \nu_p}\right)\mathrm{d}x^\mu \wedge \mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_p} \\ &= \frac{1}{p!}\frac{\partial}{\partial x^\mu}\mathrm{Tr}\omega_{\nu_1 \ldots \nu_p}\mathrm{d}x^\mu \wedge \mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_p} \\ &= \mathrm{d}\left(\mathrm{Tr}\omega\right) \end{aligned}$$

This is just a reflection of the fact that d and Tr act over different algebras.

In particular, we have then:

$$\mathrm{d}\mathrm{Tr}\left(F \wedge F\right) = \mathrm{Tr}d\left(F \wedge F\right)$$

where for F is a 2-form, we have by anti-derivation that:

$$\mathrm{d}\left(F \wedge F\right) = \mathrm{d}F \wedge F + F \wedge \mathrm{d}F$$

Now, by the Bianchi identity, we have $\mathrm{d}F = F \wedge A - A \wedge F$, which gives us that:

$$\begin{aligned} \mathrm{d}\left(F \wedge F\right) &= (F \wedge A - A \wedge F) \wedge F + F \wedge (F \wedge A - A \wedge F) \\ &= F \wedge A \wedge F - A \wedge F \wedge F + F \wedge F \wedge A - F \wedge A \wedge F \\ &= -A \wedge F \wedge F + F \wedge F \wedge A \\ &= \frac{1}{4}\left(-A_\alpha \cdot F_{\beta\gamma} \cdot F_{\delta\epsilon} + F_{\alpha\beta} \cdot F_{\gamma\delta} \cdot A_\epsilon\right) \\ &\quad \cdot \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta \wedge \mathrm{d}x^\epsilon \end{aligned}$$

where again, the "$\cdot$" denotes matrix product. Now, taking the trace give us:

$$\begin{aligned} \mathrm{Tr}d\left(F \wedge F\right) &= -\frac{1}{4}\mathrm{Tr}\left(A_\alpha \cdot F_{\beta\gamma} \cdot F_{\delta\epsilon}\right)\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta \wedge (\mathrm{d}x^\epsilon) \\ &\quad + \frac{1}{4}\mathrm{Tr}\left(F_{\alpha\beta} \cdot F_{\gamma\delta} \cdot A_\epsilon\right)\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta \wedge \mathrm{d}x^\epsilon \end{aligned}$$

Now, we can simplify by making use of the property that the Trace operation is invariant under cyclic permutations of the matrix arguments. Hence:

$$\mathrm{Tr}\left(F_{\alpha\beta} \cdot F_{\gamma\delta} \cdot A_\epsilon\right) = \mathrm{Tr}\left(A_\epsilon \cdot F_{\alpha\beta} \cdot F_{\gamma\delta}\right)$$

and the second term in equation 11 may be rewritten as:

$$\begin{aligned}&\frac{1}{4}\mathrm{Tr}\left(F_{\alpha\beta}\cdot F_{\gamma\delta}\cdot A_\epsilon\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\wedge\mathrm{d}x^\epsilon\\&=\frac{1}{4}\mathrm{Tr}\left(A_\epsilon\cdot F_{\alpha\beta}\cdot F_{\gamma\delta}\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\wedge\mathrm{d}x^\epsilon\\&=\end{aligned}$$

$$\begin{aligned}&\tfrac{1}{4}\mathrm{Tr}\left(F_{\alpha\beta}\cdot F_{\gamma\delta}\cdot A_\epsilon\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\wedge\mathrm{d}x^\epsilon\\&=\tfrac{1}{4}\mathrm{Tr}\left(A_\epsilon\cdot F_{\alpha\beta}\cdot F_{\gamma\delta}\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\wedge\mathrm{d}x^\epsilon\\&=\tfrac{1}{4}\mathrm{Tr}\left(A_\alpha\cdot F_{\beta\gamma}\cdot F_{\delta\epsilon}\right)\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\wedge\mathrm{d}x^\epsilon\wedge\mathrm{d}x^\alpha\\&=\tfrac{1}{4}\mathrm{Tr}\left(A_\alpha\cdot F_{\beta\gamma}\cdot F_{\delta\epsilon}\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\wedge\mathrm{d}x^\epsilon\end{aligned}$$

As such, the second term in equation 11 is exactly the negative of the first term. Hence, we have that:

$$\mathrm{Trd}\left(F\wedge F\right)=0=\mathrm{dTr}\left(F\wedge F\right)$$

which shows that $\mathrm{Tr}\left(F\wedge F\right)$ is a closed form.

Part (viii)

We wish to show that $\mathrm{Tr}\left(F\wedge F\right)=\mathrm{dTr}\left(A\wedge\mathrm{d}A+\frac{2}{3}A\wedge A\wedge A\right)$. (This further shows that $\mathrm{Tr}\left(F\wedge F\right)$ is exact, and hence closed.)

From before, we have that $\mathrm{dTr}\left(A\wedge\mathrm{d}A+\frac{2}{3}A\wedge A\wedge A\right)=\mathrm{Trd}\left(A\wedge\mathrm{d}A+\frac{2}{3}A\wedge A\wedge A\right)$. Hence, we wish to show that:

$$\mathrm{Tr}\left(F\wedge F\right)=\mathrm{Tr}\,\mathrm{d}\left(A\wedge\mathrm{d}A+\frac{2}{3}A\wedge A\wedge A\right)$$

Consider first that:

$$\begin{aligned}F\wedge F&=\left(\mathrm{d}A+A\wedge A\right)\wedge\left(\mathrm{d}A+A\wedge A\right)\\&=\mathrm{d}A\wedge\mathrm{d}A+A\wedge A\wedge\mathrm{d}A+\mathrm{d}A\wedge A\wedge A\\&\quad+A\wedge A\wedge A\wedge A\end{aligned}$$

so that:

$$\begin{aligned}\mathrm{Tr}\left(F\wedge F\right)&=\mathrm{Tr}\left(\mathrm{d}A\wedge\mathrm{d}A\right)\\&\quad+\mathrm{Tr}\left(A\wedge A\wedge\mathrm{d}A\right)+\mathrm{Tr}\left(\mathrm{d}A\wedge A\wedge A\right)\\&\quad+\mathrm{Tr}\left(A\wedge A\wedge A\wedge A\right)\end{aligned}$$

Consider then that:

$$\begin{aligned}\mathrm{Tr}\left(A\wedge A\wedge\mathrm{d}A\right)&=\mathrm{Tr}\left(A_\alpha\cdot A_\beta\cdot\partial_\gamma A_\delta\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\\&=\mathrm{Tr}\left(\partial_\gamma A_\delta\cdot A_\alpha\cdot A_\beta\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\\&=\mathrm{Tr}\left(\partial_\alpha A_\beta\cdot A_\gamma\cdot A_\delta\right)\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\wedge\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\\&=\mathrm{Tr}\left(\partial_\alpha A_\beta\cdot A_\gamma\cdot A_\delta\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\\&=\mathrm{Tr}\left(\mathrm{d}A\wedge A\wedge A\right)\end{aligned}$$

A similar derivation shows also that $\mathrm{Tr}\left(\mathrm{d}A\wedge A\wedge A\right)=-\mathrm{Tr}\left(A\wedge\mathrm{d}A\wedge A\right)$. That is

$$\begin{aligned}\mathrm{Tr}\left(\mathrm{d}A\wedge A\wedge A\right)&=\mathrm{Tr}\left(\partial_\alpha A_\beta A_\gamma A_\delta\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\\&=\mathrm{Tr}\left(\partial_\beta A_\gamma A_\delta A_\alpha\right)\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\wedge\mathrm{d}x^\alpha\\&=-\mathrm{Tr}\left(\partial_\beta A_\gamma A_\delta A_\alpha\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\\&=-\mathrm{Tr}\left(A_\alpha\partial_\beta A_\gamma A_\delta\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\\&=-\mathrm{Tr}\left(A\wedge\mathrm{d}A\wedge A\right)\end{aligned}$$

It's important to note that despite the appearance of trace, we cannot cyclically permute the arguments at will; there can be minus signs.

$$\mathrm{Tr}\left(A\wedge A\wedge\mathrm{d}A\right)=-\mathrm{Tr}\left(A\wedge\mathrm{d}A\wedge A\right)=\mathrm{Tr}\left(\mathrm{d}A\wedge A\wedge A\right)\tag{12}$$

so that in particular:

$$\mathrm{Tr}\left(A\wedge A\wedge\mathrm{d}A\right)+\mathrm{Tr}\left(\mathrm{d}A\wedge A\wedge A\right)=2\mathrm{Tr}\left(\mathrm{d}A\wedge A\wedge A\right)$$

Meanwhile:

$$\begin{aligned}\mathrm{Tr}\left(A\wedge A\wedge A\wedge A\right)&=\mathrm{Tr}\left(A_\alpha A_\beta A_\gamma A_\delta\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\\&=\mathrm{Tr}\left(A_\beta A_\gamma A_\delta A_\alpha\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\\&=\mathrm{Tr}\left(A_\alpha A_\beta A_\gamma A_\delta\right)\mathrm{d}x^\delta\wedge\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\\&=-\mathrm{Tr}\left(A_\alpha A_\beta A_\gamma A_\delta\right)\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\wedge\mathrm{d}x^\delta\\&=-\mathrm{Tr}\left(A\wedge A\wedge A\wedge A\right)\end{aligned}$$

so that $\mathrm{Tr}\left(A\wedge A\wedge A\wedge A\right)=0$. As such, we have that:

$$\mathrm{Tr}\left(F\wedge F\right)=\mathrm{Tr}\left(\mathrm{d}A\wedge\mathrm{d}A\right)+2\mathrm{Tr}\left(\mathrm{d}A\wedge A\wedge A\right)\tag{13}$$

On the other hand:

$$\begin{aligned}&\mathrm{d}\left(A\wedge\mathrm{d}A+\tfrac{2}{3}A\wedge A\wedge A\right)\\&\quad=\mathrm{d}A\wedge\mathrm{d}A+\tfrac{2}{3}\mathrm{d}A\wedge A\wedge A\\&\qquad-\tfrac{2}{3}A\wedge\mathrm{d}A\wedge A+\tfrac{2}{3}A\wedge A\wedge\mathrm{d}A\end{aligned}$$

so that:

$$\begin{aligned}\mathrm{Tr}\left(\ldots\right)&=\mathrm{Tr}\left(\mathrm{d}A\wedge\mathrm{d}A\right)+\frac{2}{3}\mathrm{Tr}\left(\mathrm{d}A\wedge A\wedge A\right)\\&\quad-\frac{2}{3}\mathrm{Tr}\left(A\wedge\mathrm{d}A\wedge A\right)+\frac{2}{3}\mathrm{Tr}\left(A\wedge A\wedge\mathrm{d}A\right)\end{aligned}$$

In light of equation 12 however, this simpliy reduces to:

$$\begin{aligned}\mathrm{Tr}\left(\ldots\right)&=\mathrm{Tr}\left(\mathrm{d}A\wedge\mathrm{d}A\right)+\frac{6}{3}\mathrm{Tr}\left(\mathrm{d}A\wedge A\wedge A\right)\\&=\mathrm{Tr}\left(\mathrm{d}A\wedge\mathrm{d}A\right)+2\mathrm{Tr}\left(\mathrm{d}A\wedge A\wedge A\right)\end{aligned}\tag{14}$$

Comparing equations 13 and 14, we see that:

$$\mathrm{Tr}\left(F\wedge F\right)=\mathrm{Tr}\,\mathrm{d}\left(A\wedge\mathrm{d}A+\frac{2}{3}A\wedge A\wedge A\right)$$