1 Counting Indices

Part (a)

Let $V$ be a $d$-dimensional vector space. Let $A = \bigwedge^p V$ be the space of all skew-symmetric $(p, 0)$ tensors (a.k.a. covariant tensors in $p$ indices). $A$ is spanned by the elements $e_{\mu_1} \wedge e_{\mu_2} \wedge \ldots \wedge e_{\mu_p}$, where $\mu_i$ runs from 1 to $d$.

Consider that any such element containing a repeated index, e.g. $\mu_i = \mu_j$ for some $i, j$, is automatically zero due to skew-symmetry; consider also any two such elements containing the same set of indices (perhaps in different order) are related to each other by factors of $-1$. Therefore, the set of such elements that are also linearly independent, is given by choosing $p$ indices from the set of $d$ dimensions. That is, we can think of specifying a basis element by filling in the blanks:

$$e_{\ldots \mu} \wedge e_{\ldots \nu} \wedge \ldots \wedge e_{\ldots \kappa}$$

with numbers from 1 through $d$, with no repeats. Therefore, the total number of all such combinations is:

$$\dim(A) = \binom{d}{p} = \frac{d!}{p!(d-p)!}$$

Alternatively, we can specify an ordering of our indices $\mu_i$ s.t. $\mu_i > \mu_j$ for $i < j$, and we will see that there are only $\binom{d}{p}$ number of such allowed sequences.

(In the symmetric case below, we relax our strict inequality $\mu_i > \mu_j$ to allow for repeat indices, so $\mu_i \geq \mu_j$).

Part (b)

Let $S = \bigotimes^p V = \text{sym}^p V$ be the space of all symmetric $(p, 0)$ tensors. $S$ is spanned by the elements $e_{\mu_1} \otimes e_{\mu_2} \otimes \ldots \otimes e_{\mu_p}$.

In this case, repeat indices are allowed. However, any two elements containing the same set of indices (perhaps in different order) are still linearly dependent.

To consider only those elements that are linearly independent, we can specify an ordering by restricting our set of indices to those s.t. $\mu_i \geq \mu_j$ for $i < j$. Then it becomes a simple matter of counting sequences whose entries (chosen from 1 to $d$) are non-decreasing.

In particular, there’s a standard combinatorial trick for enumerating such a sequence (see “multiset coefficient”). Given a basis element $e_{\mu_1} \otimes e_{\mu_2} \otimes \ldots \otimes e_{\mu_p}$, we can illustrate the sequence $\mu_1, \mu_2, \ldots, \mu_p$ as a way of putting $p$ elements into $d$ boxes: the $k^{\text{th}}$ box contains as many elements as the number of times that $k$ appears in the sequence $\mu_1, \ldots, \mu_p$.

For example, let $p = 4$, $d = 6$, and consider the element $e_1 \otimes e_3 \otimes e_3 \otimes e_4$. The corresponding sequence is $1, 3, 3, 4$. 1 appears once, 2 appears no times, 3 appears twice, 4 appears once, 5 none, 6 none. Our “box” diagram is then:

```
 1 2 3 4 5 6
```

or equivalently, if we imagine putting barriers “|” between different boxes, our sequence can be depicted as:

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| | | · · · | · | · · · |
```

where the number of dots in the $k^{\text{th}}$ box represents the number of times that $k$ appears in the sequence (we omit the left barrier of box 1 and the right barrier of box $d$ - those boxes will always be fixed in their positions, so there’s no combinatorial purpose in drawing them out).

As another example, suppose we have $p = 10$, $d = 6$, and the element $e_1 \otimes e_2 \otimes e_2 \otimes e_2 \otimes e_4 \otimes e_5 \otimes e_5 \otimes e_6 \otimes e_6 \otimes e_8$. This is the sequence $1, 2, 2, 2, 4, 5, 5, 6, 6, 6$; so our box diagram is:

```
· | | | · · · | · | | · · · |
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As can be seen, we have a $1 - 1$ mapping between basis elements of $\text{sym}^p V$, and box diagrams containing $p$ dots and $d - 1$ barriers. Therefore, counting basis elements is the same as counting configurations of box diagrams. However, a box diagram is nothing more than a way of distributing $p$ dots and $d - 1$ barriers. In particular we can imagine choosing, from $p + d - 1$ slots, $p$ slots with which to fill with a dot “|” (and the rest with barriers “|”). The problem therefore is reduced to choosing $p$ slots from $p + d - 1$ slots. The number of ways for doing so is thus $\binom{p + d - 1}{p}$.

As such, the number of independent basis elements is $\binom{p + d - 1}{p}$. Hence:

$$\dim(\text{sym}^p V) = \binom{p + d - 1}{p} = \frac{(p + d - 1)!}{p!(d - 1)!}$$

2 Quantum Entanglement

Part (a)

Let $V_1$ be an $n$-dimensional vector space with basis $\{e_1^{(1)}, \ldots, e_n^{(1)}\}$. Let $V_2$ be an $m$-dimensional vector space with basis $\{e_1^{(2)}, \ldots, e_m^{(2)}\}$.

Consider that a general rank 2 tensor is of form $a = a^{ij} e_i^{(1)} \otimes e_j^{(2)}$. We desire to see if 3 vectors $x \in V_1$, and $y \in V_2$, such that $x \otimes y = a$; in other words, such that $a^{ij} = x^i y^j$. 
Suppose \( x, y \) to be vectors such that \( x \otimes y = a \). Since \( a \) has \( nm \) components, while \( x \) and \( y \) give us only \( n + m \) components, it would seem that in order to have a solution for \( x \) and \( y \), a must itself satisfy \( nm - (n + m) \) conditions, so we would have as many variables as do equations. However, in actuality a must satisfy \( nm - (n + m - 1) = (n - 1)(m - 1) \) conditions, which can be seen as follows.

Fix \( p \) and \( q \), such that \( a^{pq} \neq 0 \) (if such an entry cannot be found, then that means a was identically zero, in which case we can just take \( x = 0 \) and \( y = 0 \) as one particular trivial solution to \( x \otimes y = a \), showing that our system is decomposable). Then, supposing that \( x^i y^j = a^{ij} \) is a solution, we will have that \( x^i y^j \neq 0 \).

Given the \( n + m - 1 \) entries \( \{ x^i y^j \}_{i,j=1}^m \cup \{ x^i y^j \}_{i=1}^n \) (though \( x^i y^j \) occurs twice, it counts for only once, hence the \( n + m - 1 \) as opposed to \( n + m \)), it can be seen that the rest of the entries, \( x^i y^j, \) for \( i \neq p \) and \( j \neq q \), are completely determined as:

\[
x^i y^j = \frac{x^i y^q}{x^p y^q} x^p y^j
\]

In otherwords, we have a condition on \( a \), that for \( i \neq p \) and \( j \neq q \) we must have:

\[
a^{ij} = \frac{a^{iq}}{a^{pj}} a^{pq}
\]

Consequently, in order for the equation \( a^{ij} = x^i y^j \) to be self-consistent, we can only freely specify the numbers \( a^{pq}, j = 1, \ldots, m, \) and \( a^{iq}, i = 1, \ldots, n; \) the rest of the coefficients \( a^{ij} \) are completely determined. Therefore, we can only freely specify \( n + m - 1 \) components.

As such, the number of conditions that \( a \) must satisfy is given by \( nm - (n + m - 1) = (n - 1)(m - 1) \).

**Part (b)**

Supposing \( x, y \) to be a solution for \( a = x \otimes y \). Then we have that \( a^{ij} = x^i y^j \). Consequently, \( \forall a^{ij}, a^{kl}, a^{il}, a^{kj} \) we must have that:

\[
a^{ij} a^{kl} = \left( x^i y^j \right) \left( x^k y^l \right)
\]

\[
= \left( x^i y^q \right) \left( x^p y^j \right)
\]

\[
= a^{il} a^{kj}
\]

So we have that \( a^{ij} a^{kl} = a^{il} a^{kj}, \) or \( a^{ij} a^{kl} - a^{il} a^{kj} = 0. \) This can be succinctly expressed as the condition that:

\[
det \begin{bmatrix} a^{ij} & a^{il} \\ a^{kj} & a^{kl} \end{bmatrix} = 0
\]

That is, if \( \exists x \in V_1, y \in V_2 \) s.t. \( a = x \otimes y, \) then det \( \begin{bmatrix} a^{ij} & a^{il} \\ a^{kj} & a^{kl} \end{bmatrix} = 0 \) \( \forall i,j. \)

**Part (c)**

It is clear that condition det \( \begin{bmatrix} a^{ij} & a^{il} \\ a^{kj} & a^{kl} \end{bmatrix} = 0 \) is necessary (because if it were not true, we cannot possibly have that \( a^{ij} = x^i y^j, \) etc.). In fact, this condition is also sufficient; but it imposes upon us \( n^2 m^2 \) equations. We can equivalently relax this to only the \( (n-1)(m-1) \) equations mentioned in part (a).

We will now prove that the following three statements are equivalent:

1. \( \det \begin{bmatrix} a^{ij} & a^{il} \\ a^{kj} & a^{kl} \end{bmatrix} = 0, \forall i,j,k,l. \)
2. \( \det \begin{bmatrix} a^{pq} & a^{pl} \\ a^{kq} & a^{kl} \end{bmatrix} = 0, \forall i \neq p, \forall j \neq q. \)
3. \( \exists x \in V_1, y \in V_2, \) s.t. \( a^{ij} = x^i y^j, \forall i,j. \)

Clearly \( 1 \implies 2 \) (since if it’s true \( \forall i,j,k,l, \) then it’s true in particular for \( i = p, j = q \)). From Part (b) we have just shown that \( 3 \implies 1. \) Therefore, if we can show \( 2 \implies 3, \) then we are done.

To show that \( 2 \implies 3, \) we simply need to show that a solution \( x^i, y^j \) exists. Given det \( \begin{bmatrix} a^{pq} & a^{pj} \\ a^{qi} & a^{ij} \end{bmatrix} = 0, \) we have that:

\[
a^{ij} = \frac{a^{pq} a^{ij}}{a^{pq}}
\]

(Recall that \( a^{pq} \) was chosen to be nonzero, so that dividing by it is valid). These are exactly the \((m-1)(n-1)\) self-consistency conditions of Part (a). Therefore, it only remains to find a solution to the equations:

\[
x^i y^q = a^{iq} \quad i \neq p \]

\[
x^q y^j = a^{qj} \quad j \neq q \]

\[
x^p y^q = a^{pq}
\]

This is a set of \( m + n - 1 \) equations in \( m + n \) variables, so a solution always exists. In our case, it can be explicitly found as:

\[
x^i = \frac{a^{iq}}{a^{pq}} x^p
\]

\[
y^j = \frac{a^{pj}}{a^{pq}} y^q
\]

\[
x^p y^q = a^{pq}
\]

Note that the solution exists, but is not unique (we have a family of solutions specified by the curve \( x^p y^q = a^{pq} \); however, this is to be expected, as it is a reflection of the fact that if \( x, y \) is a solution to \( x \otimes y = a, \) then so is \( x' = \lambda x, y' = \frac{1}{\lambda} y. \)

As such, we have shown that a solution exists; so \( 2 \implies 3, \) and we are done.

### 3 Symmetric Integration

**Part (a) - 4 indices**

Consider, in \( n \)-dimensions, the integral:

\[
I_{\alpha \beta \gamma \delta} = \int_{\mathbb{R}^n} \frac{d^nk}{(2\pi)^n} k_{\alpha} k_{\beta} k_{\gamma} k_{\delta} f(k^2)
\]

First, we show that \( I_{\alpha \beta \gamma \delta} \) is numerically invariant under the action of \( o \in O(n). \) Let \( O_\alpha \) be the standard representation of \( o. \) Then we wish to show that:

\[
I_{\alpha \beta \gamma \delta} = O_{\alpha \alpha} O_{\beta \beta} O_{\gamma \gamma} O_{\delta \delta} I_{\alpha \beta \gamma \delta}
\]
Consider then:

\[ O_{\alpha\alpha}O_{\beta\beta}O_{\gamma\gamma}O_{\delta\delta}I_{\alpha\beta\gamma\delta} = \int_{\mathbb{R}^n} \frac{d^n k}{(2\pi)^n} y_\alpha y_\beta y_\gamma y_\delta f (k^2) \]

where \( y_\alpha = O_{\alpha\alpha} k_\alpha \). We now wish to convert the expression into an integral in \( y \); therefore, we make the change of variables \( y_\alpha = O_{\alpha\alpha} k_\alpha \) for every \( k \). For \( o \in \mathcal{O} (n) \), then \( k^2 = y^2 \). Likewise, we’ll have:

\[ \frac{d^n k}{(2\pi)^n} = \det \left( \frac{\partial (y)}{\partial (k)} \right) \frac{d^n y}{(2\pi)^n} \]

where \( \det \left( \frac{\partial (y)}{\partial (k)} \right) \) is the Jacobian. Since our transformation was given by \( y_\alpha = O_{\alpha\alpha} k_\alpha \), then we have:

\[ \frac{\partial y_\alpha}{\partial k_\beta} = O_{\alpha\alpha} \delta_{\alpha\beta} = O_{\alpha\beta} \]

So the entries of the Jacobian matrix \( \frac{\partial (y)}{\partial (k)} \) are exactly \( \left( \frac{\partial (y)}{\partial (k)} \right)_{\alpha\beta} = O_{\alpha\beta} \). Now, for \( o \in \mathcal{O} (n) \), then \( \det \left( \frac{\partial (y)}{\partial (k)} \right) = \pm 1 \). However, our integration domain \( \mathbb{R}^n \) will also be changed into \( (\mathbb{R}^n)^{\prime} \); note that if \( \det (o) = 1 \), then the transformation is orientation preserving, so \( \int_{(\mathbb{R}^n)^{\prime}} = \int_{\mathbb{R}^n} \); and if \( \det (o) = -1 \), then \( \int_{(\mathbb{R}^n)^{\prime}} = - \int_{\mathbb{R}^n} \). The final result therefore is that \( \int_{(\mathbb{R}^n)^{\prime}} \det \left( \frac{\partial (y)}{\partial (k)} \right) \frac{d^n y}{(2\pi)^n} = \int_{\mathbb{R}^n} \frac{d^n x}{(2\pi)^n} \)

As such, we have that:

\[ \int_{\mathbb{R}^n} \frac{d^n k}{(2\pi)^n} y_\beta y_\gamma y_\delta f (k^2) = \int_{\mathbb{R}^n} \frac{d^n y}{(2\pi)^n} y_\alpha y_\beta y_\gamma y_\delta f (y^2) \]

so \( I_{\alpha\beta\gamma\delta} \) is numerically invariant under the action of \( \mathcal{O} (n) \). Citing then formula (10.87) of the lecture notes, we have that:

\[ I_{\alpha\beta\gamma\delta} = a \delta_{\alpha\beta} \delta_{\gamma\delta} + b \delta_{\alpha\gamma} \delta_{\beta\delta} + c \delta_{\alpha\delta} \delta_{\beta\gamma} \]

Now we must determine the constants \( a, b, c \).

Set \( \beta = \alpha \neq \gamma = \delta \). This gives us:

\[ I_{\alpha\alpha\gamma\gamma} = a \delta_{\alpha\alpha} \delta_{\gamma\gamma} + b \delta_{\alpha\gamma} \delta_{\alpha\gamma} + c \delta_{\alpha\gamma} \delta_{\gamma\alpha} \]  
\[ = a \]

This gives us that:

\[ a = \int_{\mathbb{R}^n} \frac{d^n y}{(2\pi)^n} (y_\alpha)^2 (y_\gamma)^2 f (y^2) \]

without explicitly evaluating this integral, however, we can still observe that we will have \( a = b = c \), by complete symmetry of \( I_{\alpha\alpha\gamma\gamma} = I_{\alpha\gamma\alpha\gamma} = I_{\alpha\gamma\gamma\alpha} \) (since \( I_{\alpha\beta\gamma\delta} \) is completely symmetric). Therefore, we will have:

\[ I_{\alpha\beta\gamma\delta} = a \cdot (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}) \]

Finally, suppose we contract \( \alpha \) with \( \beta \), and \( \gamma \) with \( \delta \). This gives us:

\[ I_{\alpha\alpha\gamma\gamma} = a \cdot (n^2 + n + n) \]  
\[ = a \cdot n (n + 2) \]

However, this integral is also:

\[ I_{\alpha\alpha\gamma\gamma} = \sum \sum \int_{\mathbb{R}^n} \frac{d^n y}{(2\pi)^n} (y_\alpha)^2 (y_\gamma)^2 f (y^2) \]

\[ = \int_{\mathbb{R}^n} \frac{d^n y}{(2\pi)^n} (y^2)^2 f (y^2) \]

\[ = A \]

So we have that \( A = a \cdot n (n + 2) \). As such, \( a = \frac{A}{n(n+2)} \), and we will have:

\[ I_{\alpha\beta\gamma\delta} = \frac{A}{n(n+2)} \cdot (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}) \]

**Part (b) - 5 indices**

Consider now the quantity:

\[ I_{\alpha\beta\gamma\delta} = \int_{\mathbb{R}^n} \frac{d^n k}{(2\pi)^n} k_\alpha k_\beta k_\gamma k_\delta f (k^2) \]

The integral \( I_{\alpha\beta\gamma\delta} \) is likewise preserved under \( o \in \mathcal{O} (n) \); however, since it has rank 5, an odd number, then we cannot express it as a linear combinations of \( \delta \)-Kroneckers, which will always have even rank. As such, we must have that \( I_{\alpha\beta\gamma\delta} = 0 \).

That this is true in our case can be alternatively verified as follows: If any index occurs an odd number of times, then the integrand is anti-symmetric in that variable; and since our limits of integration is symmetric, then our integral will vanish. However, it can be seen that there is no way, in rank 5, to have all indices occur an even number of times. Therefore, the integrand is always anti-symmetric, and so the integral is zero.

## 4 Leonardo da Vinci’s Problem II

Given a material of cylindrical cross section \( \Gamma \in \mathbb{R}^2 \) in the \( xy \) plane, with a centroid \( O \) as the origin of our coordinate system, defined s.t.:

\[ \int_{\Gamma} x dx dy = 0 \quad \int_{\Gamma} y dx dy = 0 \]

Consider now a deformation, in the neighborhood of \( O \), of the form:

\[ \eta^x = - \frac{\sigma}{R} xy \]
\[ \eta^y = \frac{\sigma}{2R} x^2 - \frac{\sigma}{2R} y^2 - \frac{1}{2R} z^2 \]
\[ \eta^z = \frac{1}{R} z^2 \]

with the coordinates deforming as \( \tilde{x}^2 = x^2 + \eta^x \).

We wish to analyze the effect of this deformation on the centroid, as a function of \( z \). The easiest way to see that the centroid deflects as the curve \( y = -\frac{1}{2R} z^2 \) is to simply note that, in \( \eta^y \), the terms which depend on \( z \) is just \(-\frac{1}{2R} z^2 \).

Alternatively, we can see that this is true in a pseudo-analytic manner. Consider a neighborhood \( N = (-\varepsilon, \varepsilon) \times (-\delta, \delta) \) around the original centroid \( O \). Let \((x_0, y_0)\) be the coordinates of the new
So for this equation to be true to order 2, we must have that $x_c = 0$, hence our centroid only deflects in the $yz$ plane.

For $y_c$, we proceed similarly:

$$0 = \int_N \left( y + \frac{\sigma}{2R} x^2 - \frac{\sigma}{2R} y^2 - \frac{1}{R} y^2 - y_c \right) dxdy$$

$$= \int_N y dxdy + \frac{\sigma}{2R} \int_N x^2 dxdy - \frac{\sigma}{2R} \int_N y^2 dxdy$$

$$- \frac{1}{2R} \int_N z^2 dxdy - y_c \int_N dxdy$$

$$= \frac{\sigma}{R} (\epsilon \delta) - \frac{\sigma}{R} (\epsilon \delta^3) - \frac{4}{2R} \epsilon^2 \delta^2 - 4y_c \delta^2$$

So to order 2, our equation reads:

$$0 = -4 \frac{y_c}{2R} \epsilon^2 \delta^2 - 4y_c \delta^2$$

hence we must have:

$$y_c = -\frac{1}{2R} \epsilon^2$$

This is the trajectory of our centroid subject to the above deformations. It can be seen that for small $\frac{\sigma}{2R}$, our trajectory $y_c = -\frac{1}{2R} \epsilon^2$ is the approximation to:

$$y = (R^2 - \epsilon^2)^{1/2} - R$$

(because for $y = (R^2 - z^2)^{1/2} - R$, we have approximately $y \approx \frac{-\epsilon^2}{2R} - \frac{\epsilon^4}{8R} + ...$, which is a semi-circle of radius $R$). Alternatively, we can directly apply the curvature formula:

$$\frac{1}{R_{\text{curvature}}} = \frac{|y''|}{(1 + y'^2)^{3/2}} \approx \frac{|y''|}{(1 + y^2)^{3/2}}; \text{ for small } y'$$

and we see that our radius of curvature gives us $R_{\text{curvature}} = R$.

We wish now to construct the strain tensor. First, we calculate the necessary derivatives:

$$\frac{\partial y''}{\partial y} = -\frac{\sigma}{4} y, \quad \frac{\partial y''}{\partial y} = -\frac{\sigma}{4} x, \quad \frac{\partial y''}{\partial y} = 0$$

Now we can calculate the strain tensor. By symmetry, it is only necessary to calculate 6 of the 9 components:

$$\begin{align*}
\sigma_{xx} &= -\frac{\sigma}{2} y \\
\sigma_{xy} &= 0 \\
\sigma_{x} &= -\frac{\sigma}{2} y \\
\sigma_{xx} &= 0 \\
\sigma_{yx} &= 0 \\
\sigma_{zz} &= 0 \\
\end{align*}$$

Finally, we wish to calculate the stress tensor. By the generalized Hooke’s law, we have that:

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$$

Note that $e_{kk} = (-2\frac{\sigma}{R} + \frac{\lambda}{R}) y$, we have that:

$$\sigma_{ij} = \frac{\lambda}{R} \delta_{ij} (1 - 2\sigma) y + 2\mu e_{ij}$$

Since $\delta_{ij}$ only has diagonal terms, and since $e_{ij}$ was seen to also have only diagonal terms, we find that:

$$\begin{align*}
\sigma_{ij} &= 0 \text{ if } i \neq j \\
\sigma_{xx} &= (\lambda (1 - 2\sigma) - 2\mu \sigma) \frac{y}{R} \\
\sigma_{yy} &= (\lambda (1 - 2\sigma) - 2\mu \sigma) \frac{y}{R} \\
\sigma_{zz} &= (\lambda (1 - 2\sigma) + 2\mu) \frac{y}{R} \\
\end{align*}$$

Using, from the book, the equation that:

$$Y = \lambda (1 - 2\sigma) + 2\mu$$

This gives us that:

$$\begin{align*}
\lambda (1 - 2\sigma) - 2\mu \sigma &= \lambda (1 - 2\sigma) - Y + 2\mu \\
&= \lambda (1 - 2\sigma) + 2\mu - Y \\
&= Y - Y = 0 \\
\end{align*}$$

so we have:

$$\begin{align*}
\sigma_{xx} &= 0 \\
\sigma_{yy} &= 0 \\
\sigma_{zz} &= Y \frac{y}{R} \\
\end{align*}$$

That is,

$$\begin{align*}
\sigma_{ij} &= \begin{cases} 
Y \frac{y}{R} & \text{if } i = j = z \\
0 & \text{else}
\end{cases}
\end{align*}$$

From this, we can deduce that the deformation satisfies the force-free surface boundary condition. Let the energy of the beam be defined as:

$$E = \frac{1}{2} \int_{\text{Beam}} e_{ij} c_{ijkl} c_{kld} d^3 x$$
Noting that $\varepsilon_{ijkl}\varepsilon_{kl} = \sigma_{ij}$ is just the generalized Hooke's law (in its most general form), and so we have that:

$$\mathcal{E} = \frac{1}{2} \iiint_{\text{Beam}} \varepsilon_{ij} \sigma_{ij} d^3x$$

Note also that since $\sigma_{ij}$ is only non-zero for $i = j = z$, then our only non-zero term in the contraction is:

$$\varepsilon_{ij} \sigma_{ij} = \varepsilon_{zz} \sigma_{zz} = \frac{y}{R} \frac{y}{R} Y$$

this gives us:

$$\mathcal{E} = \frac{1}{2} \frac{Y}{R^2} \iiint_{\text{Beam}} y^2 d^3x$$

Note finally that we can decompose our volume Beam as:

$$\text{Beam} = \Gamma \times \gamma$$

where $\gamma$ is the curve $y = -\frac{x^2}{a^2}$ of the centroid deformation in the $yz$ plane; and $\Gamma$ is the just the cross section in the $xy$ plane. This allows us to write:

$$\mathcal{E} = \frac{1}{2} \frac{Y}{R^2} \int_{\gamma} y^2 dy \int_{\gamma} d\gamma$$

where $I = \int_{\gamma} y^2 dy$. Finally, by the approximation made earlier that, for $\frac{dy}{dx} \ll q$, we have $\frac{1}{R} \approx \frac{d^2y}{dx^2}$ and $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \approx 1$, this gives us that:

$$\mathcal{E} = \frac{1}{2} \frac{Y}{R^2} \int_{\gamma} \left(\frac{d^2y}{dx^2}\right)^2 dx$$

5 Maxwell Stress

Let the Maxwell stress tensor be:

$$\Pi_{ij} = \varepsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} |E|^2 \right) + \mu_0 \left( H_i H_j - \frac{1}{2} \delta_{ij} |H|^2 \right)$$

In a linear dielectric, we have that $D_i = \varepsilon_0 E_i$ and $H_i = \frac{1}{\mu_0} B_i$. Therefore, we equivalently have that:

$$\Pi_{ij} = \varepsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} |E|^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} |B|^2 \right)$$

Note that $\Pi_{ij} = \Pi_{ji}$, as expected.

For $\mathbf{E}, \mathbf{B}$ satisfying Maxwell's equations, we wish to show that:

$$\partial_s \Pi_{ij} = \left( \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} \right)_j + \frac{d}{dt} \left( \frac{1}{c^2} (\mathbf{E} \times \mathbf{H})_j \right)$$

or in terms of components:

$$\partial_s \Pi_{ij} = \rho E_j + \varepsilon_{jik} J_i B_k + \frac{d}{dt} \left( \frac{1}{c^2} \varepsilon_{jik} E_i H_k \right)$$

That is:

$$\partial_s \Pi_{ij} - \rho E_j = \varepsilon_{jik} J_i B_k + \varepsilon_0 \varepsilon_{jik} \frac{d}{dt} (E_i B_k)$$

(1)

Note that Maxwell's equation, in terms of components, read:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad \partial_s E_a = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \partial_s B_a = 0$$

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \quad \Rightarrow \quad \varepsilon_{iab} \partial_s E_b = -\frac{dB_i}{dt}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{d\mathbf{E}}{dt} \quad \Rightarrow \quad \varepsilon_{iab} \partial_s B_a = \mu_0 J_i + \varepsilon_0 \mu_0 \frac{dE_i}{dt}$$

First, we wish to evaluate $\partial_s \Pi_{ij}$. To this this, we can consider $\partial_s \Pi_{ij}$, and then contract over $s$ and $i$. Now, we have that:

$$\partial_s \Pi_{ij} = \partial_s \left( \varepsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} |E|^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} |B|^2 \right) \right)$$

$$= \varepsilon_0 \partial_s (E_i E_j) - \frac{1}{2} \varepsilon_0 \delta_{ij} \partial_s (|E|^2)$$

$$+ \frac{1}{\mu_0} \partial_s (B_i B_j) - \frac{1}{2 \mu_0} \delta_{ij} \partial_s (|B|^2)$$

Note that:

$$\partial_s (|E|^2) = \partial_s (E_a E_a) = \partial_s E_a E_a + E_a (\partial_s E_a)$$

$$= 2 (\partial_s E_a) E_a$$

Likewise, we have that:

$$\partial_s (|B|^2) = 2 (\partial_s B_a) B_a$$

Therefore, we will have:

$$\partial_s \Pi_{ij} = \varepsilon_0 \left( \partial_s E_i E_j + \varepsilon_0 E_i (\partial_s E_j) \right)$$

$$- \varepsilon_0 \delta_{ij} (\partial_s E_a) E_a$$

$$+ \frac{1}{\mu_0} \left( \partial_s B_i B_j + \frac{1}{\mu_0} B_i (\partial_s B_j) \right)$$

$$- \frac{1}{2 \mu_0} \delta_{ij} (\partial_s B_a) B_a$$

Now, we contract over $s$ and $i$:

$$\partial_s \Pi_{ij} = \varepsilon_0 (\partial_s E_a) E_j + \varepsilon_0 E_a (\partial_s E_j)$$

$$- \varepsilon_0 \delta_{ij} (\partial_s E_a) E_a$$

$$+ \frac{1}{\mu_0} \left( \partial_s B_a B_j + \frac{1}{\mu_0} B_a (\partial_s B_j) \right)$$

$$- \frac{1}{2 \mu_0} \delta_{ij} (\partial_s B_a) B_a$$

Note that terms:

$$\partial_s E_a = \frac{\rho}{\varepsilon_0}$$

$$\partial_s B_a = 0$$

are exactly two of Maxwell’s equations (eqn. [2] & [3]): additionally:

$$\delta_{ij} (\partial_s E_a) = \partial_s E_a$$

$$\delta_{ij} (\partial_s B_a) = \partial_s B_a$$

Therefore, this gives us:

$$\partial_s \Pi_{ij} - \rho E_j = \varepsilon_0 E_a (\partial_s E_j) - \varepsilon_0 (\partial_s E_a) E_a$$

$$+ \frac{1}{\mu_0} B_a (\partial_s B_j) - \frac{1}{\mu_0} (\partial_s B_a) B_a$$

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Now from (eqn 1), if we can show that:

\[
\varepsilon_{ijk} J_i B_k + \varepsilon_0 \varepsilon_{ijk} \frac{d}{dt} (E_i B_k) = \varepsilon_0 E_a (\partial_a E_j) - \varepsilon_0 (\partial_j E_a) E_a + \frac{1}{\mu_0} B_a (\partial_a B_j) - \frac{1}{\mu_0} (\partial_j B_a) B_a
\]

Then we are done.

Consider that:

\[
\varepsilon_0 \varepsilon_{ijk} \frac{d}{dt} (E_i B_k) = \varepsilon_0 \varepsilon_{ijk} \frac{dE_i}{dt} B_k + \varepsilon_0 \varepsilon_{ijk} \frac{dB_k}{dt} = \varepsilon_0 \varepsilon_{ijk} \left( \frac{1}{\mu_0} \varepsilon_{ijk} \partial_a B_b - \frac{1}{\varepsilon_0} J_j \right) B_k - \varepsilon_0 \varepsilon_{ijk} \varepsilon_{kab} \partial_b E_a
\]

where in the last equality we made use of Maxwell's equations (eqn. 4 & 5). Expanding, we obtain:

\[
\varepsilon_0 \varepsilon_{ijk} \frac{d}{dt} (E_i B_k) = \varepsilon_0 \varepsilon_{ijk} \varepsilon_{iab} \left( \partial_a B_b \right) B_k - \varepsilon_0 \varepsilon_{ijk} \varepsilon_{kab} E_i \left( \partial_a E_b \right)
\]

or:

\[
\varepsilon_0 \varepsilon_{ijk} \frac{d}{dt} (E_i B_k) + \varepsilon_{ijk} J_k B_k = \varepsilon_0 \varepsilon_{ijk} \varepsilon_{iab} \left( \partial_a B_b \right) B_k - \varepsilon_0 \varepsilon_{ijk} \varepsilon_{kab} E_i \left( \partial_a E_b \right)
\]

Note that:

\[
\varepsilon_{ijk} \varepsilon_{iab} = \varepsilon_{ikj} \varepsilon_{iab} = \delta_{ka} \delta_{jb} - \delta_{kb} \delta_{ja}
\]

and:

\[
\varepsilon_{ijk} \varepsilon_{kab} = \varepsilon_{kji} \varepsilon_{kab} = \delta_{ja} \delta_{ib} - \delta_{jb} \delta_{ia}
\]

so this gives us:

\[
\varepsilon_0 \varepsilon_{ijk} \frac{d}{dt} (E_i B_k) + \varepsilon_{ijk} J_k B_k = \varepsilon_0 \varepsilon_{ijk} \varepsilon_{iab} \left( \partial_a B_b \right) B_k - \varepsilon_0 \varepsilon_{ijk} \varepsilon_{kab} E_i \left( \partial_a E_b \right)
\]

Comparing this to equation 6, we see that we have achieved our desired equality, so we are done.