1) Lie Bracket Geometry: Consider the vector fields $X = y\partial_x$, $Y = \partial_y$ in $\mathbb{R}^2$. Find the flows associated with these fields, and use them to verify the statements made in the lecture about the geometric interpretation of the Lie bracket.

2) Frobenius’ theorem: Show that the pair of vector fields $L_z = x\partial_y - y\partial_x$ and $L_y = z\partial_x - x\partial_z$ in $\mathbb{R}^3$ is in involution. Show further that the general solution of the system of partial differential equations

$$(x\partial_y - y\partial_z)f = 0,$$

$$(x\partial_z - z\partial_x)f = 0,$$

in $\mathbb{R}^3$ is $f(x, y, z) = F(x^2 + y^2 + z^2)$, where $F$ is an arbitrary function.

3) Rolling ball: In class we mentioned the rolling conditions for a ball on a table:

$$\dot{x} = \psi \sin \theta \sin \phi + \dot{\theta} \cos \phi,$$

$$\dot{y} = -\psi \sin \theta \cos \phi + \dot{\theta} \sin \phi, \quad (*)$$

$$0 = \dot{\psi} \cos \theta + \dot{\phi}.$$

Here, we are using the “Y” convention for Euler angles. In this convention $\theta$ and $\phi$ are the usual spherical polar co-ordinate angles with respect to the space-fixed $xyz$ axes. They specify the direction of the body-fixed $Z$ axis about which we make the final $\psi$ rotation.

Euler angles: we first rotate the ball through an angle $\phi$ about the $z$ axis, thus taking $y \rightarrow Y'$, then through $\theta$ about $Y''$, and finally through $\psi$ about $Z$, so taking $Y'' \rightarrow Y$. 

1
a) Show that \((\ast)\) are indeed the no-slip rolling conditions
\[
\begin{align*}
\dot{x} &= \omega_y, \\
\dot{y} &= -\omega_x, \\
0 &= \omega_z,
\end{align*}
\]
where \((\omega_x, \omega_y, \omega_z)\) are the components of the ball’s angular velocity in the \(xyz\) space-fixed frame.

b) Solve the three constraints \((\ast)\) so as to obtain the vector fields
\[
\begin{align*}
\text{roll}_x &= \partial_x - \sin \phi \cot \theta \partial_\phi + \cos \phi \partial_\theta + \csc \theta \sin \phi \partial_\psi, \\
\text{roll}_y &= \partial_y + \cos \phi \cot \theta \partial_\phi + \sin \phi \partial_\theta - \csc \theta \cos \phi \partial_\psi.
\end{align*}
\]

c) Show that
\[
[\text{roll}_x, \text{roll}_y] = -\text{spin}_z,
\]
where \(\text{spin}_z \equiv \partial_\phi\), corresponds to a rotation about a vertical axis through the point of contact. This is a new motion, being forbidden by the \(\omega_z = 0\) condition.

d) Show that
\[
[\text{spin}_x, \text{roll}_x] = \text{spin}_x, \\
[\text{spin}_y, \text{roll}_y] = \text{spin}_y,
\]
where the new vector fields
\[
\begin{align*}
\text{spin}_x &\equiv - (\text{roll}_y - \partial_y), \\
\text{spin}_y &\equiv (\text{roll}_x - \partial_x),
\end{align*}
\]
correspond to rotations of the ball about the space-fixed \(x\) and \(y\) axes through its centre, and with the centre of mass held fixed.

We have generated five independent vector fields from the original two. Therefore, by sufficient rolling to-and-fro, we can position the ball anywhere on the table, and in any orientation.

4) Killing Vector: The metric on the unit sphere equipped with polar co-ordinates is
\[
g(\ , \ ) = d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi.
\]
Consider
\[
V_x = - \sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi,
\]
the vector field of a rigid rotation about the \(x\) axis. Compute the Lie derivative \(\mathcal{L}_{V_x} g\), and show that it is zero.