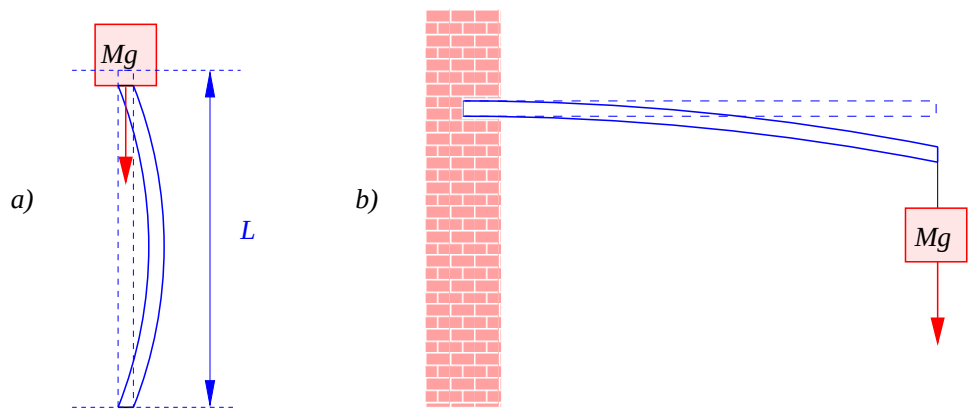
Physics 508 | Prof. M. Stone
Handout 3 | **Mathematical Methods in Physics I** | 2117 ESB
Fall 2017 | HOMEWORK 3 | University of Illinois

1) Elastic Rods. The elastic energy per unit length of a bent steel rod is given by $\frac{1}{2}YI/R^2$. Here R is the radius of curvature due to the bending, Y is the Young's modulus of the steel and $I = \iint y^2 dxdy$ is the moment of inertia of the rod's cross section about an axis through its centroid and perpendicular to the plane in which the rod is bent. If the rod is only slightly bent into the yz plane and lies close to the z axis, show that this elastic energy can be approximated as

$$U[y] = \int_0^L \frac{1}{2} YI \left(y''\right)^2 \, dz,$$

where the prime denotes differentiation with respect to z and L is the length of the rod. We will use this approximate energy functional to discuss two practical problems.



a) Euler's problem: The buckling of a slender column. The rod is used as a column which supports a compressive load Mg directed along the z axis (which is vertical). Show that when the rod buckles slighly (*i.e.* bends with both ends remaining on the z axis) the total energy, including the gravitational potential energy of the loading mass M, can be approximated by

$$U[y] = \int_0^L \left\{ \frac{YI}{2} \left(y''\right)^2 - \frac{Mg}{2} \left(y'\right)^2 \right\} dz.$$

By considering deformations of the form

$$y(z) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi z}{L}$$

show that the column is unstable to buckling and collapse once $Mg \geq \frac{\pi^2}{L^2} YI$.

b) Leonardo da Vinci's problem: The light cantilever. Here we take the z axis as horizontal and the y axis as being vertical. The rod is used as a beam or cantilever and is fixed into a wall so that $y(0) = 0 = y'(0)$. A weight Mg is hung from the end $z = L$ and the beam sags in the $-y$ direction. Your task is find $y(z)$ for $0 < z < L$. You may ignore the weight of the beam itself.

- Write down the complete expression for the energy, including the gravitational potential energy of the weight.
- Find the differential equation and boundary conditions at $z = 0$, L that arise from minimizing the total energy. In doing this take care not to throw away any term arising from the integration by parts.You may find the following identity to be of use:

$$\frac{d}{dz}(f'g'' - fg''') = f''g'' - fg''''.$$

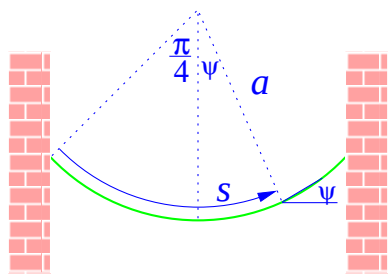
- Solve the equation. You should find that the displacement of the end of the beam is $y(L) = -\frac{1}{3}\frac{MgL^3}{YI}$.

2) Lagrange multipliers: Express the function $f(x, y) = 13x^2 + 8xy + 7y^2$ as $\mathbf{x}^t\mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a 2×2 matrix and $\mathbf{x}^t = (x, y)$. By solving the associated eigenvalue problem, use the method of undetermined multipliers to find the stationary points and values of the function $f(x, y) = 13x^2 + 8xy + 7y^2$ on the circle $x^2 + y^2 = 1$.

3) The Catenary Again: We can describe a catenary curve in parametric form as $x(s)$, $y(s)$, where s is the arc-length. The potential energy is then simply $\int_0^L \rho g y(s) ds$ where ρ is the mass per unit length of the hanging chain. The x, y are not independent functions of s, however, because $\dot{x}^2 + \dot{y}^2 = 1$ at every point on the curve. (What do the dots mean, and why is this so?)

a) Introduce infinitely many Lagrange multipliers $\lambda(s)$ to enforce this constraint, one for each point s on the curve. From the resulting functional derive two coupled equations for the catenary, one for $x(s)$ and one for $y(s)$. By thinking about the forces acting on a small section of the cable, and perhaps by introducing the angle ψ where $\dot{x} = \cos\psi$ and $\dot{y} = \sin\psi$ (so that s and ψ are *intrinsic coordinates* for the curve) interpret these equations and relate $\lambda(s)$ to the position-dependent tension $T(s)$ in the chain.

b) You are provided with a light-weight line of length $\pi a/2$ and some lead shot of total mass M. By using equations from the previous part (suitably modified to take into account the position dependent $\rho(s)$) or otherwise, determine how the lead should be distributed along the line if the loaded line is to hang in an arc of a circle of radius a when its ends are attached to two points at the same height.



Weighted line.