1) Critical Mass: An infinite slab of fissile material has thickness $L$. The neutron density $n(r)$ in the material obeys the equation
\[
\frac{\partial n}{\partial t} = D \nabla^2 n + \lambda n + \mu,
\]
where $n$ is zero at the surface of the slab at $x = 0, L$. Here $D$ is the neutron diffusion constant, the term $\lambda n$ describes the creation of new neutrons by induced fission, and $\mu$ is the rate of production per unit volume of neutrons by spontaneous fission. Assume that $n$ depends only on $x$ and $t$, and that $\lambda$ and $\mu$ are constants.

a) Expand both $n$ and $\mu$ as series
\[
n(x, t) = \sum_m a_m(t) \varphi_m(x), \quad \mu = \sum_m b_m \varphi_m(x),
\]
where the $\varphi_m$ are a complete orthonormal set of functions you think suitable for solving the problem.

b) Find an explicit expression for the coefficients $a_m(t)$ in terms of their initial values $a_m(0)$.

c) Determine the critical thickness, $L_{\text{crit}}$, above which the slab will explode.

d) Assuming that $L < L_{\text{crit}}$, find the equilibrium distribution $n_{eq}(x)$ of neutrons in the slab. (You may either sum your series expansion to get an explicit closed-form answer, or use another (Green function?) method.)

2) Semi-infinite Rod: Consider the heat equation
\[
\frac{\partial \theta}{\partial t} = D \nabla^2 \theta, \quad 0 < x < \infty
\]
with the temperature $\theta(x, t)$ obeying the initial condition $\theta(x, 0) = \theta_0$ for $0 < x < \infty$, and the boundary condition $\theta(0, t) = 0$.

a) Show that the boundary condition at $x = 0$ can be satisfied at all times by introducing a suitable mirror image of the initial data in the region $-\infty < x < 0$, and then applying the heat kernel for the entire real line to this extended initial data. Show that the solution of the semi-infinite rod problem can be expressed in terms of the error function
\[
\text{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi.
\]

b) Solve the same problem by using a Fourier integral expansion in terms of $\sin kx$ on the half-line $0 < x < \infty$ and obtaining the time evolution of the Fourier coefficients. Invert
3) 2-D Electron Gas:

A two-dimensional gas of electrons is confined at the $z = 0$ interface between two semi-infinite dielectric slabs. Each slab has dielectric constant $\varepsilon$. A perturbation of the electron charge-density propagates as a wave through the electron gas. The surface-charge density on the interface is therefore given by $\sigma(x, t) = \sigma_0 + \sigma_1(x, t)$, where $\sigma_0$ is constant and the small-amplitude perturbation $\sigma_1$ takes the form

$$\sigma_1(x, t) = a \exp\{i(kx - \omega t)\}.$$ 

Assume that electrons act as classical particles of mass $m$ with local velocity,

$$v(x, t) = v_0 \exp\{i(kx - \omega t)\},$$

and that the only significant force is due to the electric field produced by the charge density perturbation.

a) Use Laplace’s equation

$$-\nabla^2 \phi = \varepsilon^{-1} \delta(z) \sigma(x, t)$$

to find the electrical potential $\phi(x, z, t)$ due to the charge.

b) From $\phi(x, z, t)$ find the electric field component $E_x(x, z = 0, t)$ parallel to and within the electron gas, and hence the acceleration $\partial v(x, t)/\partial t$ of the electrons.

c) Linearize the charge continuity equation

$$\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma v}{\partial x} = 0,$$

and use it to relate $a$ and $v_0$. Hence show that the dispersion equation relating the frequency $\omega$ to the wavenumber $k$ is

$$\omega^2 = \gamma |k|.$$ 

Express the coefficient $\gamma$ in terms of $m$, $\varepsilon$, $\sigma_0$ and the electron charge $q = -e$. 

4) **Seasonal Heat Waves:** Suppose that the measured temperature of the air above the arctic permafrost is expressed as a Fourier series

\[
\theta(t) = \theta_0 + \sum_{n=1}^{\infty} \theta_n \cos n\omega t,
\]

where \( T = \frac{2\pi}{\omega} \) is one year. Solve the heat equation for the soil temperature

\[
\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial z^2}, \quad 0 < z < \infty
\]

with this boundary condition, and find the temperature \( \theta(z, t) \) at a depth \( z \) below the surface as a function of time. Observe that the sub-surface temperature fluctuates with the same period as that of the air, but with a phase lag that depends on the depth. Also observe that the longest period temperature fluctuations penetrate the deepest into the ground. (Hint: for each Fourier component, write \( \theta \) as Re\([A_n(z) \exp i\omega t]\) where \( A_n \) is a complex function of \( z \).)