1) Test functions and distributions:
   a) Let $f(x)$ be a smooth function.
      i) Show that $f(x)\delta(x) = f(0)\delta(x)$. Deduce that
         $$\frac{d}{dx}[f(x)\delta(x)] = f'(x)\delta(x).$$
      ii) We might also have used the product rule to conclude that
         $$\frac{d}{dx}[f(x)\delta(x)] = f'(x)\delta(x) + f(x)\delta'(x).$$
         By integrating both against a test function, show this expression for the derivative of $f(x)\delta(x)$ is equivalent to that in part i).
   b) Let $G(x)$ be a smooth function that decreases rapidly to zero as $|x| \to \infty$, and $\varphi(x)$ a smooth function such that its derivative $\varphi'(x)$ decreases rapidly to zero as $|x| \to \infty$. Show that
         $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi'(x)\varphi'(y)G(|x-y|) \, dx \, dy = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\varphi(x) - \varphi(y)\right]^2 G''(|x-y|) \, dx \, dy.$$
   c) Let $\varphi(x)$ be a test function. Using the definition of the principal part integral, show that
         $$\frac{d}{dt} \left\{ P \int_{-\infty}^{\infty} \frac{\varphi(x)}{x-t} \, dx \right\} = P \int_{-\infty}^{\infty} \frac{\varphi(x) - \varphi(t)}{(x-t)^2} \, dx$$
         To do this fix the value of the cutoff $\epsilon$ and then differentiate the resulting $\epsilon$-regulated integral, taking care to include the terms arising from the $t$ dependence of the limits at $x = t \pm \epsilon$.

2) One-dimensional scattering theory: Consider the one-dimensional Schrödinger equation
   $$-\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$
   where $V(x)$ is zero except in a finite interval $[-a, a]$ near the origin.
Let $L$ denote the left asymptotic region, $-\infty < x < -a$, and similarly let $R$ denote $\infty > x > a$. For $E = k^2$ and $k > 0$ there will be scattering solutions of the form

$$
\psi_k(x) = \begin{cases} 
eikx + r_L(k)e^{-ikx}, & x \in L, \\
t_L(k)e^{ikx}, & x \in R,
\end{cases}
$$

describing waves incident on the potential $V(x)$ from the left. For $k < 0$ there will be solutions with waves incident from the right

$$
\psi_k(x) = \begin{cases} t_R(k)e^{ikx}, & x \in L, \\
eikx + r_R(k)e^{-ikx}, & x \in R,
\end{cases}
$$

The wavefunctions in $[-a,a]$ will naturally be more complicated. Observe that $[\psi_k(x)]^*$ is also a solution of the Schrödinger equation.

By using properties of the Wronskian, show that:

a) $|r_{L,R}|^2 + |t_{L,R}|^2 = 1,$

b) $t_L(k) = t_R(-k).$

c) Deduce from parts a) and b) that $|r_L(k)| = |r_R(-k)|.$

d) Take the specific example of $V(x) = \lambda\delta(x - b)$ with $|b| < a$. Compute the transmission and reflection coefficients and hence show that $r_L(k)$ and $r_R(-k)$ may differ by a phase.

3) Reduction of Order: Sometimes additional information about the solutions of a differential equation enables us to reduce the order of the equation, and so solve it.

a) Suppose that we know that $y_1 = u(x)$ is one solution to the equation

$$
y'' + V(x)y = 0.
$$

By trying $y = u(x)v(x)$ show that

$$
y_2 = u(x)\int^x d\xi \frac{d\xi}{u^2(\xi)}
$$

is also a solution of the differential equation. Is this new solution ever merely a constant multiple of the old solution, or must it be linearly independent? (Hint: evaluate the Wronskian $W(y_2, y_1).$)

b) Suppose that we are told that the product, $y_1y_2$, of the two solutions to the equation $y'' + p_1y' + p_2y = 0$ is a constant. Show that this requires $2p_1p_2 + p_2' = 0.$

c) By using ideas from part b) or otherwise, find the general solution of the equation

$$(x + 1)x^2y'' + xy' - (x + 1)^3y = 0.$$

4) Normal forms and the Schwarzian derivative: We saw in class that if $y$ obeys a second-order linear differential equation

$$
y'' + p_1y' + p_2y = 0
$$
then we can make always make a substitution $y = w\tilde{y}$ so that $\tilde{y}$ obeys an equation without a first derivative:

$$\tilde{y}'' + q(x)\tilde{y} = 0.$$ 

Suppose $\psi(x)$ obeys a Schrödinger equation

$$\left(-\frac{1}{2} \frac{d^2}{dx^2} + [V(x) - E]\right) \psi = 0.$$ 

a) Make a smooth and invertible change of independent variable by setting $x = x(z)$ and find the second order differential equation in $z$ obeyed by $\psi(z) \equiv \psi(x(z))$. Find the $\tilde{\psi}(z)$ that obeys an equation with no first derivative. Show that this equation is

$$\left(-\frac{1}{2} \frac{d^2}{dz^2} + (x')^2[V(x(z)) - E] - \frac{1}{4}\{x, z\}\right) \tilde{\psi}(z) = 0,$$

where the primes denote differentiation with respect to $z$, and

$$\{x, z\} \equiv \frac{x'''}{x'} - \frac{3}{2} \left(\frac{x''}{x'}\right)^2$$

is called the *Schwarzian* derivative of $x$ with respect to $z$. Schwarzian derivatives play an important role in conformal field theory and string theory.

b) Now combine a sequence of maps $x \rightarrow z \rightarrow w$ to establish *Cayley’s identity*

$$\left(\frac{dz}{dw}\right)^2 \{x, z\} + \{z, w\} = \{x, w\}.$$ 

(Hint: If this takes you more than a line or two, you are missing the point of the problem.)