1) Test functions and distributions:

a) Let \( f(x) \) be a smooth function.

i) Show that \( f(x) \delta(x) = f(0) \delta(x) \). Deduce that

\[
\frac{d}{dx} [f(x) \delta(x)] = f(0) \delta'(x).
\]

ii) We might also have used the product rule to conclude that

\[
\frac{d}{dx} [f(x) \delta(x)] = f'(x) \delta(x) + f(x) \delta'(x).
\]

By integrating both against a test function, show this expression for the derivative of \( f(x) \delta(x) \) is equivalent to that in part i).

b) Let \( G(x) \) be a smooth function that decreases rapidly to zero as \( |x| \to \infty \), and \( \varphi(x) \) a smooth function such that its derivative \( \varphi'(x) \) decreases rapidly to zero as \( |x| \to \infty \).

Show that

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi'(x) \varphi'(y) G(|x-y|) \, dx \, dy = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\varphi(x) - \varphi(y)|^2 G''(|x-y|) \, dx \, dy.
\]

c) In a paper\(^1\) that has recently been cited in the literature on topological insulators a distribution \( \delta^{(1/2)}(x) \) is defined by setting

\[
\delta^{(1/2)}(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k|^{1/2} e^{ikx}.
\]

The Fourier transform on the RHS is clearly divergent, so we need to decide how to interpret it. Let’s try to define the evaluation of \( \delta^{(1/2)} \) on a test function \( \varphi(x) \) as

\[
\int_{-\infty}^{\infty} \delta^{(1/2)}(x) \varphi(x) \, dx \overset{\text{def}}{=} \lim_{\mu \to 0^+} \left\{ \int_{-\infty}^{\infty} \delta^{(1/2)}_{\mu}(x) \varphi(x) \, dx \right\},
\]

where

\[
\delta^{(1/2)}_{\mu}(x) \overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{ikx} |k|^{1/2} e^{-\mu |k|} \frac{dk}{2\pi}
\]

\[
= \sqrt{\frac{1}{4\pi}} (x^2 + \mu^2)^{-3/4} \cos \left( \frac{3}{2} \tan^{-1} \left( \frac{x}{\mu} \right) \right).
\]

(Could you have evaluated this integral if I had not given you the answer?)

Plot some graphs of $\delta_{\mu}^{(1/2)}(x)$ for various values of $\mu$, and so get an idea of how it behaves as the convergence factor $e^{-\mu |k|} \to 1$. Deduce that

$$\int_{-\infty}^{\infty} \delta_{\mu}^{(1/2)}(x) \varphi(x) \, dx = -\sqrt{\frac{1}{8\pi}} \int_{-\infty}^{\infty} \frac{1}{|x|^{3/2}} \{ \varphi(x) - \varphi(0) \} \, dx. $$

(Hint: Observe that $\delta_{\mu}^{(1/2)}(x)$ is the Fourier transform of a function that vanishes at $k = 0$. What property of the the graph of $\delta_{\mu}^{(1/2)}(x)$ does this imply?)

d) Let $\varphi(x)$ be a test function. Using the definition of the principal part integral, show that

$$\frac{d}{dt} \left\{ P \int_{-\infty}^{\infty} \frac{\varphi(x)}{(x-t)} \, dx \right\} = P \int_{-\infty}^{\infty} \frac{\varphi(x) - \varphi(t)}{(x-t)^2} \, dx $$

To do this fix the value of the cutoff $\epsilon$ and then differentiate the resulting $\epsilon$-regulated integral, taking care to include the terms arising from the $t$ dependence of the limits at $x = t \pm \epsilon$.

2) **One-dimensional scattering theory**: Consider the one-dimensional Schrödinger equation

$$-\frac{d^2 \psi}{dx^2} + V(x) \psi = E \psi $$

where $V(x)$ is zero except in a finite interval $[-a, a]$ near the origin.

Let $L$ denote the left asymptotic region, $-\infty < x < -a$, and similarly let $R$ denote $\infty > x > a$. For $E = k^2$ and $k > 0$ there will be scattering solutions of the form

$$\psi_k(x) = \begin{cases} e^{ikx} + r_L(k) e^{-ikx}, & x \in L, \\ t_L(k) e^{ikx}, & x \in R, \end{cases} $$

describing waves incident on the potential $V(x)$ from the left. For $k < 0$ there will be solutions with waves incident from the right

$$\psi_k(x) = \begin{cases} t_R(k) e^{ikx}, & x \in L, \\ e^{ikx} + r_R(k) e^{-ikx}, & x \in R. \end{cases} $$

The wavefunctions in $[-a, a]$ will naturally be more complicated. Observe that $[\psi_k(x)]^*$ is also a solution of the Schrödinger equation.

By using properties of the Wronskian, show that:
a) \(|r_{L,R}|^2 + |t_{L,R}|^2 = 1,
\)
b) \(t_{L}(k) = t_{R}(-k).
\)
c) Deduce from parts a) and b) that \(|r_{L}(k)| = |r_{R}(-k)|.
\)
d) Take the specific example of \(V(x) = \lambda \delta(x - b)\) with \(|b| < a\). Compute the transmission and reflection coefficients and hence show that \(r_{L}(k)\) and \(r_{R}(-k)\) may differ by a phase.

3) Reduction of Order: Sometimes additional information about the solutions of a differential equation enables us to reduce the order of the equation, and so solve it.

a) Suppose that we know that \(y_1 = u(x)\) is one solution to the equation
\[
y'' + V(x)y = 0.
\]
By trying \(y = u(x)v(x)\) show that
\[
y_2 = u(x) \int x \frac{d\xi}{u^2(\xi)}
\]
is also a solution of the differential equation. Is this new solution ever merely a constant multiple of the old solution, or must it be linearly independent? (Hint: evaluate the Wronskian \(W(y_2, y_1)\).)

b) Suppose that we are told that the product, \(y_1y_2\), of the two solutions to the equation \(y'' + p_1y' + p_2y = 0\) is a constant. Show that this requires \(2p_1p_2 + p'_2 = 0\).

c) By using ideas from part b) or otherwise, find the general solution of the equation
\[
(x + 1)x^2y'' + xy' - (x + 1)^3y = 0.
\]

4) Normal forms and the Schwarzian derivative: We saw in class that if \(y\) obeys a second-order linear differential equation
\[
y'' + p_1y' + p_2y = 0
\]
then we can make always make a substitution \(y = w\bar{y}\) so that \(\bar{y}\) obeys an equation without a first derivative:
\[
\bar{y}'' + q(x)\bar{y} = 0.
\]
Suppose \(\psi(x)\) obeys a Schrödinger equation
\[
\left( -\frac{1}{2} \frac{d^2}{dx^2} + [V(x) - E] \right) \psi = 0.
\]
a) Make a smooth and invertible change of independent variable by setting \( x = x(z) \) and find the second order differential equation in \( z \) obeyed by \( \psi(z) \equiv \psi(x(z)) \). Find the \( \tilde{\psi}(z) \) that obeys an equation with no first derivative. Show that this equation is

\[
\left( -\frac{1}{2} \frac{d^2}{dz^2} + (x')^2 [V(x(z)) - E] - \frac{1}{4} \{x, z\} \right) \tilde{\psi}(z) = 0,
\]

where the primes denote differentiation with respect to \( z \), and

\[
\{x, z\} \equiv \frac{x'''}{x'} - \frac{3}{2} \left( \frac{x''}{x'} \right)^2
\]

is called the Schwarzian derivative of \( x \) with respect to \( z \). Schwarzian derivatives play an important role in conformal field theory and string theory.

b) Now combine a sequence of maps \( x \rightarrow z \rightarrow w \) to establish Cayley’s identity

\[
\left( \frac{dz}{dw} \right)^2 \{x, z\} + \{z, w\} = \{x, w\}.
\]

(Hint: If this takes you more than a line or two, you are missing the point of the problem.)