1) **Missing State**: In Homework Set 4 you found that the Schrödinger equation

\[
\left(-\frac{d^2}{dx^2} - 2 \text{sech}^2 x\right) \psi = E \psi
\]

has eigensolutions

\[
\psi_k(x) = e^{ikx}(-ik + \tanh x)
\]

with eigenvalue \( E = k^2 \).

- Show that for \( x \) large and positive \( \psi_k(x) \approx A e^{ikx}e^{i\delta(k)} \), while for \( x \) large and negative \( \psi_k(x) \approx A e^{ikx}e^{-i\delta(k)} \), the complex constant \( A \) (which you must write down explicitly) being the same in both cases. Express \( \delta(k) \) as the inverse tangent of an algebraic expression in \( k \).

- Impose periodic boundary conditions \( \psi(-L/2) = \psi(+L/2) \) where \( L \gg 1 \). Find the allowed values of \( k \) and hence an explicit expression for the \( k \)-space density, \( \rho(k) = \frac{dn}{dk} \), of the eigenstates.

- Compare your formula for \( \rho(k) \) with the corresponding expression, \( \rho_0(k) = L/2\pi \), for the eigenstate density of the zero-potential equation and compute the integral

\[
\Delta N = \int_{-\infty}^{\infty} \{\rho(k) - \rho_0(k)\}dk.
\]

- Deduce that one eigenfunction has gone missing from the continuum and presumably become a localized bound state. (You will have found an explicit expression for this localized eigenstate in Homework Set 4.)

2) **Continuum Completeness**: Consider the differential operator

\[
\hat{L} = -\frac{d^2}{dx^2}, \quad 0 \leq x < \infty
\]

with self-adjoint boundary conditions \( \psi(0)/\psi'(0) = \tan \theta \) for some fixed angle \( \theta \).

- Show that when \( \tan \theta < 0 \) there is a single normalizable negative-eigenvalue eigenfunction localized near the origin, but none when \( \tan \theta > 0 \).

- Show that there is a continuum of positive-eigenvalue eigenfunctions of the form \( \psi_k(x) = \sin(kx + \delta(k)) \) where the phase shift \( \delta \) is found from

\[
e^{i\delta(k)} = \frac{1 + ik \tan \theta}{\sqrt{1 + k^2 \tan^2 \theta}}.
\]
Write down (no justification required) the appropriate completeness relation

\[ \delta(x - x') = \int \frac{dn}{dk} N_k \psi_k(x) \psi_k(x') \, dk + \sum_{\text{bound}} \psi_n(x) \psi_n(x') \]

with an explicit expression for the product (not the separate factors) of the density of states and the normalization constant \( N_k \), and with the correct limits on the integral over \( k \).

Confirm that the \( \psi_k \) continuum on its own, or together with the bound state when it exists, form a complete set. You will do this by evaluating the integral

\[ I(x, x') = \frac{2}{\pi} \int_0^\infty \sin(kx + \delta(k)) \sin(kx' + \delta(k)) \, dk \]

and interpreting the result. You will need the following standard integral

\[ \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \frac{1}{1 + k^2 t^2} = \frac{1}{2|t|} e^{-|x|/|t|}. \]

To get full credit, you must show how the bound state contribution switches on and off with \( \theta \). The modulus signs are essential for this.

3) Fredholm Alternative:

A heavy elastic bar with uniform mass \( m \) per unit length lies almost horizontally. It is supported by a distribution of upward forces \( F(x) \).

\[ U[y] = \int_0^L \left\{ \frac{1}{2} \kappa (y')^2 - (F(x) - mg)y \right\} \, dx, \]

which gives (homework 2!) the equation

\[ \hat{L} y \equiv \kappa \frac{d^4y}{dx^4} = F(x) - mg, \quad y'' = y''' = 0 \quad \text{at} \quad x = 0, L. \]

Show that the boundary conditions are such that the operator \( \hat{L} \) is self-adjoint with respect to an inner product with weight function 1.
• Find the zero modes which span the null space of $\hat{L}$.

• If there are $n$ linearly independent zero modes, then the codimension of the range of $\hat{L}$ is also $n$. Using your explicit solutions from the previous part, find the conditions that must be obeyed by $F(x)$ for a solution of $\hat{L}y = F - mg$ to exist. What is the physical meaning of these conditions?

• The solution to the equation and boundary conditions is not unique. Is this non-uniqueness physically reasonable? Explain.