Physics 487 – Homework #10 **due Friday 1 pm**

You may use anything from the **486/7 formula sheets** without derivation
… but do try to see how far you can get on your own on your Desert Island. ☺
You may also use **wolframalpha**.com or similar tool to evaluate your integrals
after you set the up in a form that can be **directly entered** into such tools.

Shorthand:

$\psi_{\mathbf{A}}$ = wavefunction for case $E > V$ = classically-**Allowed** region where E is **Above** the potential

$\psi_{\mathbf{B}}$ = wavefunction for case $E < V$ = classically-**Bonkers** regions where E is **Below** the potential

WKB solution forms in regions where they are valid (extensively discussed in class) :
with $p(x) \equiv \sqrt{2m(E - V(x))}$,

$$\psi_{\mathbf{A}}(x) = \frac{A}{\sqrt{p(x)}} \exp\left[\pm i \int^x \frac{p(x')}{\hbar} dx' \right] \qquad \text{and} \qquad \psi_{\mathbf{B}}(x) = \frac{B}{\sqrt{|p(x)|}} \exp\left[\pm \int^x \frac{|p(x')|}{\hbar} dx' \right]$$

Connection formulae to match WKB solutions at **turning points** $x = a$ (i.e. points where $E = V(a)$) :
using the shorthand $\int k \equiv \int p(x')/\hbar \; dx'$,

barrier on the LEFT:

$$\psi_{\mathbf{B}}(x) = \frac{1}{2} \frac{C}{\sqrt{|p|}} \exp\left[-\int_x^a |k| \right] \quad \text{matches to} \quad \psi_{\mathbf{A}}(x) = \frac{C}{\sqrt{p}} \cos\left[\int_a^x k - \frac{\pi}{4} \right]$$

$$\psi_{\mathbf{B}}(x) = \frac{1}{2} \frac{D}{\sqrt{|p|}} \exp\left[+\int_x^a |k| \right] \quad \text{matches to} \quad \psi_{\mathbf{A}}(x) = \frac{D}{\sqrt{p}} \cos\left[\int_a^x k + \frac{\pi}{4} \right]$$

barrier on the RIGHT:

$$\psi_{\mathbf{B}}(x) = \frac{1}{2} \frac{C}{\sqrt{|p|}} \exp\left[-\int_a^x |k| \right] \quad \text{matches to} \quad \psi_{\mathbf{A}}(x) = \frac{C}{\sqrt{p}} \cos\left[\int_x^a k - \frac{\pi}{4} \right]$$

$$\psi_{\mathbf{B}}(x) = \frac{1}{2} \frac{D}{\sqrt{|p|}} \exp\left[+\int_a^x |k| \right] \quad \text{matches to} \quad \psi_{\mathbf{A}}(x) = \frac{D}{\sqrt{p}} \cos\left[\int_x^a k + \frac{\pi}{4} \right]$$

NOTE 1 : The "barrier on the **right**" formulae are **IDENTICAL** to the "barrier on the **left**" ones except for the **reversed order of the integral bounds**. In ALL cases, the lower bound is at smaller x than the upper bound.

SUPER IMPORTANT NOTE 2 : When **NOT TO USE** the **Connection Formulae**

- ▶ The Connection Formulae above replace the standard boundary conditions "ψ & ψ′ continuous" in situations where $\boldsymbol{V(x)}$ is **continuous** i.e. **has a finite slope** at the **turning point** $x=a$.
- ▶ If $\boldsymbol{V(x)}$ is instead **discontinuous** at a **turning point** (i.e. a step, such as a switchover to a $V=\infty$ region) then you use the standard boundary conditions. Why? Because in the case of a step, there is no finite region around the turning point where $E \approx V(x)$ ∴ $p \approx 0$ ∴ $\lambda \approx \infty$, and so there is no finite transition region where the WKB approximation fails, requiring Airy functions and the Connection Formulae above. You simply have a WKB-approximated waveform to the left of the turning point, and another WKB waveform to the right, and you patch them together with continuity & continuity-of-derivative as you have always done.

Problem 1 : Bouncing Ball = Discussion 10 Question 2 *Qual Problem / bits of Griffiths 8.5 & 8.6*

Let's consider the quantum mechanical analog to the classical problem of a ball of mass m bouncing elastically on the floor.

(a) What is the potential energy, as a function of height x above the floor? (For negative x, the potential is *infinite* — the ball can't get there at all.) Sketch this $V(x)$ so you know what to do next!

▶ Aside: Can you find an exact solution of the Schrödinger equation for this potential? Remember what we learned in class about the Schrödinger equation for linear potentials? *Eeeek! (Halloween! Scary!)*

(b) Using the WKB approximation, find the allowed energies, E_n, of the bouncing ball in terms of m, g, and $\hbar$.

(c) Give the ball a mass of $m = 0.1$ kg = 100 g. (A regulation tennis ball has a mass around 60 g, so let's call it a tennis ball.) What are the energies of the ground state and first-excited state of the tennis ball, and what maximum height does the tennis ball reach in each of these states?

(d) What values do you get for question (c) if you replace the tennis ball with a neutron?

FYI: If we replaced it with an electron – a charged particle of extremely tiny mass – we would have to design *insanely perfect shielding* to shield the electron from *all* stray electromagnetic fields, or we would never see the tiny effect of the gravitational field. So … on page 333, Griffiths has a footnote with a reference to an actual experiment that actually measured the quantum effects of a neutron "bouncing" in gravity. Oy!

(e) What value of the quantum number n is required for the 100 g tennis ball to reach a maximum heigh of 1m? (*Hahahahahaha.)*

Problem 2 : WKB Hydrogen Atom *adapted from Griffiths 8.14*

Your derivation of the 1D SHO energy spectrum in discussion 10 gave you the following quantization result:

$$\boxed{\int_a^b p(x)\,dx = \left(n-\frac{1}{2}\right)\pi\hbar},$$

where a and b are the two turning points. If you look back at your work, you will see that the *form* of V(x) didn't affect the derivation of this formula at all, just the fact that it was *continuous at both turning points* and so required the Connection Formulae. Beyond its continuity, the influence of $V(x)$ on the formula is entirely in the quantity p(x). So, the above result is universal for continuous potentials V(x) in 1D (hence the box). We discussed in class how, in the WKB semi-classical limit, p(x) really *does* represent an approximate measure of the momentum at each location x. If we treat p as the particle's actual momentum, and calculate the integral of p(x) dx over a complete round trip from a to b to a = one period of a classical particle's motion in the well, we have derived the Wilson-Sommerfeld quantization rule from Old Quantum Theory:

$$\oint_{1\text{ period}} |p(x)|\,dx = \int_a^b p(x)\,dx + \int_b^a -p(x)\,dx = 2\int_a^b p(x)\,dx = 2\left(n-\frac{1}{2}\right)\pi\left(\frac{h}{2\pi}\right) = \left(n-\frac{1}{2}\right)h$$

except our rule is offset by $h/2$ from the original nh. Note that we had to play around with signs in this calculation because the "$p = -i\hbar\, d/dx$" in all our calculations is actually the x-component of $\vec{p} = -i\hbar\vec{\nabla}$.

We can also treat the radial part of the Schrödinger equation exactly like the 1D Schrödinger equation because it has the same form ... *as long as* you remember to make the substitution $u \equiv rR$, don't forget the centrifugal term in the effective potential, and stay away from $r = 0$. (See equations 4.37, 4.38 in Griffiths.) We can thus airlift our universal WKB quantization rule for continuous 1D potentials to the radial case:

$$\int_a^b p(r)\,dr = \left(n - \frac{1}{2}\right)\pi\hbar \ .$$

(Second reminder: don't forget the centrifugal = angular-momentum term in the $V(r)$ that goes into $p(r)$.)
Use this WKB result to estimate the bound state energies for hydrogen. Note that you recover the Bohr energies = the results of the exact calculation when $n >> l$ and $n >> ½$.

Problem 3 : Tunnelling via Stark effect *Griffiths 8.16*

Griffiths problem 8.16 (a,b,c,d)

▶ Part(c) involves a new quantity: the **lifetime τ** of an unstable system. (Griffiths discusses it in Example 8.2.) If a system is unstable in the sense that it can