Phys 487 Discussion 13 – Fermi’s Golden Rule

Given

\[ H(t) = H^{(0)} + H'(t), \]

\[ \{ E^{(0)}_n, |n^{(0)}\rangle \} = \text{the eigen-* of } H^{(0)} \]

\[ \text{initial state } |\psi(t=0)\rangle = |i^{(0)}\rangle \]

then

\[ |\psi(t)\rangle = \sum_n c_n(t) e^{-i\omega t} |n^{(0)}\rangle \]

\[ \text{with } i\hbar \dot{c}_f(t) = \sum_n H'_{fn} e^{i\omega_f t} c_n(t) \]

\[ \text{and to 1st order in } H' \ll H^{(0)}, \]

\[ c_f(t) \approx \delta_{fi} + \frac{1}{i\hbar} \int_0^t H'_{fi}(t') e^{i\omega_f t'} dt' \]

\[ \implies P_{i\rightarrow f} = |c_f(t)|^2 \]

Problem 1: Fermi’s Golden Rule for a constant perturbation

The simplest time-dependent perturbation is a constant potential \( V \) that just “turns on” at some time \( t = 0 \):

\[ V(t) = 0 \text{ for } t < 0 \quad \& \quad V(t) = V = \text{constant for } t \geq 0. \]

Suppose we have a system with a solvable unperturbed Hamiltonian \( H_0 \) plus the time-dependent perturbation \( V(t) \) given above. What is the transition probability \( P_{i\rightarrow f} = |c_f(t)|^2 \) to first order?

(a) Derive the following result:

\[ P_{i\rightarrow f} = \frac{|V_{\beta}|^2}{\hbar^2} \left[ \frac{\sin(\omega_{\beta} t / 2)}{\omega_{\beta} t / 2} \right]^2 t^2 \text{ for } i \neq f. \]

You will need the “half-angle formula” \( 1 - \cos \theta = 2 \sin^2 \left( \theta / 2 \right) \).

(b) Prove the following weird but important Dirac delta-function relation:

\[ \delta(ax) = \frac{\delta(x)}{a}. \]

(Think UNITS.)

(c) Prove that the following is a delta function:

\[ \lim_{\omega \rightarrow 0} \frac{1}{\pi} \frac{\sin^2(ax)}{ax^2} = \delta(x). \]

(You will need \( \int_{-\infty}^{\infty} \sin^2 x / x^2 = \pi \).)

(d) Combining the above, show that the transition rate

\[ R_{i\rightarrow f} \equiv \frac{P_{i\rightarrow f}}{t} = \frac{2\pi}{\hbar} |V_{\beta}|^2 \delta(E_f - E_i) \]

in the limit where energy is conserved, i.e. where \( E_f - E_i \rightarrow 0 \).

This relation is one form of Fermi’s Golden Rule for energy-conserving transitions.

(e) Is it reasonable to insist that energy is conserved despite the change in potential energy? Return to expression (a) and consider its dependence on \( \omega_{\beta} = \hbar(E_f - E_i) \). As you can quickly check with some sort of machine, the function \( \sin^2 x / x^2 \) is peaked at \( x = 0 \) and has a width of about \( \pi \). (It reaches \( 0.4 \approx 1/2 \) at \( x = \pm \pi/2 \).) Given this info, what range of \( \omega_{\beta} \) values keeps the transition probability \( P_{i\rightarrow f} \) within a factor of about 2 of its maximum value? Your answer will involve time, \( t \). Does the range of probable transition frequencies increase or decrease with \( t \)?

(f) Hopefully what you found was that, as \( t \rightarrow \infty \), the width in reasonably-probable transition frequencies goes to zero. This is a very important result! What does \( t \) represent, exactly? Once you know, you can say something like this

“In the limit that the perturbation \( V(t) \) _____ (words)_____, the only transition frequency with any finite probability is \( \omega_{\beta} = _____ \), which means that energy is _____ in this limit.”

We have thus clarified the conditions under which part (d) is a valid result.
(g) You just found that energy is conserved more exactly in the transition from state $i$ to state $f$ as the time $t$ that the perturbation has been ON increases. What relation involving a German name is this related to?

(h) The perturbation can never be on forever, i.e. we can never reach the limit $t \to \infty$, so there is always some non-zero range of final-state energies $E_f \approx E_i$ that can be reached from an initial-state energy $E_i$ ... but a transition $E_i \to E_f$ can only occur if there a state with energy $E_f$ actually exists. It is customary to inject information about the availability of final states into Fermi’s Golden Rule using the quantity $\rho(E_f) = \text{the density of final states} = \text{#states per unit energy}$. This quantity has units of 1/energy. The energy-conserving delta function $\delta(E_f - E_i)$ in our earlier version of Fermi’s Golden Rule also has units of 1/energy. To get the most familiar form of F.G.R., we simply replace the one-final-state-only $\delta$-function with the density of states:

$$R_{i \to f} \equiv \frac{P_{i \to f}}{t} = \frac{2\pi}{\hbar} |V_{fi}|^2 \rho(E_f) \bigg|_{E_f \approx E_i}$$

Fermi’s Golden Rule for energy-conserving transitions.