Phys 487 Discussion 10 – The WKB Approximation

Shorthand:

ψ_A = wavefunction for case $E > V$ = classically-**Allowed** region where E is **Above** the potential

ψ_B = wavefunction for case $E < V$ = classically-**Bonkers** regions where E is **Below** the potential

WKB solution forms in regions where they are valid (extensively discussed in class) :
with $p(x) \equiv \sqrt{2m(E-V(x))}$,

$$\psi_{\mathbf{A}}(x) = \frac{A}{\sqrt{p(x)}} \exp\left[\pm i \int^x \frac{p(x')}{\hbar} dx'\right] \qquad \text{and} \qquad \psi_{\mathbf{B}}(x) = \frac{B}{\sqrt{|p(x)|}} \exp\left[\pm \int^x \frac{|p(x')|}{\hbar} dx'\right]$$

Connection formulae to match WKB solutions at **turning points** $x = a$ (i.e. points where $E = V(a)$) :
using the shorthand $\int k \equiv \int p(x')/\hbar \; dx'$,

barrier on the LEFT :

$$\psi_{\mathbf{B}}(x) = \frac{1}{2} \frac{C}{\sqrt{|p|}} \exp\left[-\int_x^a |k|\right] \quad \text{matches to} \quad \psi_{\mathbf{A}}(x) = \frac{C}{\sqrt{p}} \cos\left[\int_a^x k - \frac{\pi}{4}\right]$$

$$\psi_{\mathbf{B}}(x) = \frac{1}{2} \frac{D}{\sqrt{|p|}} \exp\left[+\int_x^a |k|\right] \quad \text{matches to} \quad \psi_{\mathbf{A}}(x) = \frac{D}{\sqrt{p}} \cos\left[\int_a^x k + \frac{\pi}{4}\right]$$

barrier on the RIGHT :

$$\psi_{\mathbf{A}}(x) = \frac{C}{\sqrt{p}} \cos\left[\int_x^a k - \frac{\pi}{4}\right] \quad \text{matches to} \quad \psi_{\mathbf{B}}(x) = \frac{1}{2} \frac{C}{\sqrt{|p|}} \exp\left[-\int_a^x |k|\right]$$

$$\psi_{\mathbf{A}}(x) = \frac{D}{\sqrt{p}} \cos\left[\int_x^a k + \frac{\pi}{4}\right] \quad \text{matches to} \quad \psi_{\mathbf{B}}(x) = \frac{1}{2} \frac{D}{\sqrt{|p|}} \exp\left[+\int_a^x |k|\right]$$

NOTE 1 : The “barrier on the **right**” formulae are **IDENTICAL** to the “barrier on the **left**” ones except for the **reversed order of the integral bounds**. In ALL cases, the lower bound is at smaller *x* than the upper bound.

SUPER IMPORTANT NOTE 2 : The connection formulae replace the standard boundary conditions “ψ continuous & ψ' continuous” in situations where V(x) has a **slope** at the turning point *x=a*. If V(x) is **discontinuous** at a turning point (i.e. a step of some sort, as with a piecewise $V(x)$, or a switchover to a $V=\infty$ region) then you just use the standard boundary conditions. Why? Because in the case of a step, there is no finite region around the turning point where $E \approx V(x)$, and so there is **no transition region where the WKB approximation fails**, requiring Airy functions and the connection formulae above. You just have a WKB-approximated waveform to the left of the turning point, and another WKB waveform to the right, and you patch them together with continuity and continuity-of-derivative as you have always done.

Problem 1 : 1D SHO ™ *Griffiths 8.7*

(a) Before you do anything else, make a small sketch next to the “barrier on the LEFT” formulae above and another one next to the “barrier on the RIGHT” formulae that indicate what these situations refer to. On each sketch draw V(x) with a solid line, E with a dashed line, and mark the turning point x=a. See your lecture notes.

(b) Use the WKB approximation to find the allowed energies of the harmonic oscillator.

▶ **HINT**: Follow the procedure we used in Lecture for the infinite well … but this time you have a sloping potential, not a stepwise one, so you must use the connection formulae rather than that standard boundary conditions to match your wavefunction forms in the allowed and forbidden regions.

▶ **Power Option**: You see those two really irritating $\pi/4$ phase factors in the connection formulae? You can use this problem to *derive* their values. Call the positive one δ_p and the negative one δ_n (or whatever letters). We can agree that, by symmetry, they must have the same magnitude, so they can only differ by a sign. If you leave the $\pi/4$s as variables all the way through your calculation, then match your final result to the spectrum you KNOW for the SHO, you can "derive" the values of these phase factors. ☺

Problem 2 : Bouncing Ball *Qual Problem / hacked bits of Griffiths 8.5 & 8.6*

Now let's consider the quantum mechanical analog to the classical problem of a ball of mass m bouncing elastically on the floor.

(a) What is the potential energy, as a function of height x above the floor? (For negative x, the potential is *infinite* — the ball can't get there at all.) Sketch this $V(x)$ so you know what to do next!

▶ Aside: Can you find an exact solution of the Schrödinger equation for this potential? Remember what we learned in class about the Schrödinger equation for linear potentials? *Eeeek! (Halloween! Scary!)*

(b) Using the WKB approximation, find the allowed energies, E_n, of the bouncing ball in terms of m, g, and $\hbar$.

(c) Give the ball a mass of $m = 0.1$ kg = 100 g. (A regulation tennis ball has a mass around 60 g, so let's call it a tennis ball.) What are the energies of the ground state and first-excited state of the tennis ball, and what maximum height does the tennis ball reach in each of these states?

(d) What values do you get for question (c) if you replace the tennis ball with a neutron?

FYI: If we replaced it with an electron – a charged particle of extremely tiny mass – we would have to design *insanely perfect shielding* to shield the electron from *all* stray electromagnetic fields, or we would never see the tiny effect of the gravitational field. So … on page 333, Griffiths has a footnote with a reference to an actual experiment that actually measured the quantum effects of a neutron "bouncing" in gravity. Oy!

(e) What value of the quantum number n is required for the 100 g tennis ball to reach a maximum heigh of 1m? (*Hahahahahaha.)*