Problem 1: Time-Dependence of a Starting State

The Hamiltonian for a certain three-level system is represented by the matrix

\[
H = \begin{pmatrix}
a & 0 & b \\
0 & c & 0 \\
b & 0 & a \\
\end{pmatrix}
\]

where \(a, b,\) and \(c\) are real numbers. Assume \(a - c \neq \pm b\).

(a) We can already tell that the basis in which this Hamiltonian is written is not \{ the system’s energy eigenstates \}. How can we tell?

(b) If the system starts out in the state \(|\varphi(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\), what is its time-dependence \(|\varphi(t)\rangle\)?

(c) If the system starts out in the state \(|\varphi(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\), what is its time-dependence \(|\varphi(t)\rangle\)?

Problem 2: Too Easy, Drill Sergeant, Too Easy!

The Hamiltonian for a certain three-level system is represented by the matrix

\[
H = \hbar \omega \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \\
\end{pmatrix}
\]

Two other observables, \(A\) and \(B\), are represented by the matrices

\[
A = \lambda \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 2 \\
\end{pmatrix}
\]

and

\[
B = \mu \begin{pmatrix}
2 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}
\]

where \(\omega, \lambda,\) and \(\mu\) are positive real numbers.

(a) Find the eigenvalues and (normalized) eigenvectors of \(H, A,\) and \(B\).

(b) Suppose the system starts out in the generic state \(|\varphi(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}\) with \(|c_1|^2 + |c_2|^2 + |c_3|^2 = 1\).

Find the expectation values at \(t = 0\) of the observables \(H, A,\) and \(B\).
(c) What is \( \mathcal{A}(t) \)? If you measured the energy of this state at time \( t \), what values might you get, and what is the probability of each? Answer the same questions for observables \( A \) and \( B \) if you are feeling energetic.

**Problem 3 : A Perturbed Hamiltonian in Matrix Form**

Griffiths 6.9

Consider a quantum system with only three linearly independent states. Suppose the Hamiltonian, in matrix form, is

\[
H = V_0 \begin{pmatrix}
1 - \varepsilon & 0 & 0 \\
0 & 1 & \varepsilon \\
0 & \varepsilon & 2 \\
\end{pmatrix}
\]

where \( V_0 \) is a constant and \( \varepsilon \) is a small number << 1.

(a) Write down the eigenvectors and eigenvalues of the unperturbed Hamiltonian, i.e. the Hamiltonian you obtain by setting the small parameter \( \varepsilon \) to zero.

(b) Solve for the exact eigenvalues of \( H \) without using any perturbation-theory formulae at all. Expand each of them as a power series in \( \varepsilon \), up to second order.

(c) Use first- and second-order non-degenerate perturbation theory to find the approximate eigenvalue for the state that grows out of the non-degenerate eigenvector of \( H_0 \). The formulae are found at the bottom of this page. Compare the exact result that you found in (a).

**Problem 4 : Qual Time! A Second-Order Perturbation Theory Problem**

A particle moves in a 3D SHO with potential energy \( V(r) \). A weak perturbation \( \delta V(x,y,z) \) is applied:

\[
V(r) = \frac{m\omega^2}{2} \left( x^2 + y^2 + z^2 \right) \quad \text{and} \quad \delta V(x,y,z) = Uxyz + \frac{U^2}{\hbar \omega} x^2 y^2 z^2
\]

where \( U \) is a small parameter. Use perturbation theory to calculate the change in the ground state energy to order \( O(U^2) \). Use without proof all the results you like from the 1D SHO → see supplementary file on website.

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**Perturbation Theory Formulae**

- “zeroth-order” Hamiltonian \( H_0 \) has exact eigenvalues \( \{ E_n^{(0)} \} \) and eigenstates \( \{| n^{(0)} \rangle \} \)
- actual Hamiltonian \( H = H_0 + H' \) where \( H' \) is a small correction to \( H_0 \) (a “perturbation”, \( H' \ll H_0 \))
- series expansion of \( H \) eigenvalues: \( E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \ldots \) for each \( n \), where \( E_n^{(0)} \gg E_n^{(1)} \gg E_n^{(2)} \gg \ldots \)
- series expansion of \( H \) eigenstates: \( | n \rangle = | n^{(0)} \rangle + | n^{(1)} \rangle + | n^{(2)} \rangle + \ldots \) for each \( n \), where \( | n^{(0)} \rangle \gg | n^{(1)} \rangle \gg \ldots \)

As long as the exact eigenstates \( \{| n^{(0)} \rangle \} \) are non-degenerate and the Hamiltonian \( H = H_0 + H' \) has no explicit time-dependence, the formulae for the 1st-order and 2nd-order corrections to each energy eigenvalue \( E_n \) and energy eigenstate \( | n \rangle \) are

- \( E_n^{(1)} = \langle n^{(0)} | H' | n^{(0)} \rangle \) = the expectation value of the perturbation \( H' \) in the \( n \)th exact state,

- \( | n^{(1)} \rangle = \sum_{m \neq n} \frac{\langle m^{(0)} | H' | n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} | m^{(0)} \rangle \), \hspace{1cm} \text{and} \hspace{1cm} E_n^{(2)} = \langle n^{(0)} | H' | n^{(0)} \rangle = \sum_{m \neq n} \frac{\langle m^{(0)} | H' | n^{(0)} \rangle^2}{E_n^{(0)} - E_m^{(1)}} \).