Phys 487 Discussion 6 – Practice with Matrix Representations : Time Dependence of States & 2[nd]-Order Perturbation Theory

Problem 1 : Time-Dependence of a Starting State *Griffiths 3.37*

The Hamiltonian for a certain three-level system is represented by the matrix

$$\mathbf{H} = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}$$

where a, b, and c are real numbers. Assume $a - c \neq \pm b$.

(a) We can already tell that the **basis** in which this Hamiltonian is written is **not** { the system's energy eigenstates }. How can we tell?

(b) If the system starts out in the state $|\mathscr{S}(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, what is its time-dependence $|\mathscr{S}(t)\rangle$?

(c) If the system starts out in the state $|\mathscr{S}(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, what is its time-dependence $|\mathscr{S}(t)\rangle$?

Problem 2 : Too Easy, Drill Sergeant, Too Easy! *Griffiths 3.38*

The Hamiltonian for a certain three-level system is represented by the matrix

$$\mathbf{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Two other observables, A and B, are represented by the matrices

$$\mathbf{A} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where ω, λ, and μ are positive real numbers.

(a) Find the eigenvalues and (normalized) eigenvectors of $\mathbf{H}$, $\mathbf{A}$, and $\mathbf{B}$.

(b) Suppose the system starts out in the generic state $|\mathscr{S}(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ with $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$.

Find the expectation values at $t = 0$ of the observables H, A, and B.

(c) What is $|\mathscr{S}(t)\rangle$? If you measured the energy of this state at time t, what values might you get, and what is the probability of each? Answer the same questions for observables A and B if you are feeling energetic.

Problem 3 : A Perturbed Hamiltonian in Matrix Form *Griffiths 6.9*

Consider a quantum system with only three linearly independent states. Suppose the Hamiltonian, in matrix form, is

$$\mathbf{H} = V_0 \begin{pmatrix} (1-\varepsilon) & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & \varepsilon & 2 \end{pmatrix}$$

where V_0 is a constant and ε is a small number << 1.

(a) Write down the eigenvectors and eigenvalues of the **unperturbed Hamiltonian** , i.e. the Hamiltonian you obtain by setting the small parameter ε to zero.

(b) Solve for the *exact* eigenvalues of **H** without using any perturbation-theory formulae at all. Expand each of them as a power series in ε, up to second order.

(c) Use first- and second-order non-degenerate perturbation theory to find the approximate eigenvalue for the state that grows out of the non-degenerate eigenvector of H_0. The formulae are found at the bottom of this page. Compare the exact result that you found in (a).

Problem 4 : Qual Time! A Second-Order Perturbation Theory Problem

A particle moves in a 3D SHO with potential energy $V(r)$. A weak perturbation $\delta V(x,y,z)$ is applied:

$$V(r) = \frac{m\omega^2}{2}\left(x^2 + y^2 + z^2\right) \qquad \text{and} \qquad \delta V(x,y,z) = U\,xyz + \frac{U^2}{\hbar\omega}x^2y^2z^2$$

where U is a small parameter. Use perturbation theory to calculate the change in the ground state energy to order $O(U^2)$. Use without proof all the results you like from the 1D SHO → see supplementary file on website.

Perturbation Theory Formulae

- "zeroth-order" Hamiltonian H_0 has exact eigenvalues $\{E_n^{(0)}\}$ and eigenstates $\{|n^{(0)}\rangle\}$
- *actual* Hamiltonian $H = H_0 + H'$ where H' is a small correction to H_0 (a "perturbation", $H' \ll H_0$)
- series expansion of H eigenvalues: $E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \ldots$ for each n, where $E_n^{(0)} \gg E_n^{(1)} \gg E_n^{(2)} \gg \ldots$
- series expansion of H eigenstates: $|n\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle + |n^{(2)}\rangle + \ldots$ for each n, where $|n^{(0)}\rangle \gg |n^{(1)}\rangle \gg \ldots$

As long as the exact eigenstates $\{|n^{(0)}\rangle\}$ are **non-degenerate** and the Hamiltonian $H = H_0 + H'$ has **no explicit time-dependence**, the formulae for the 1st-order and 2nd-order corrections to each energy eigenvalue E_n and energy eigenstate $|n\rangle$ are

- $E_n^{(1)} = \langle n^{(0)}|H'|n^{(0)}\rangle$ = the expectation value of the perturbation H' in the n^{th} *exact* state,
- $|n^{(1)}\rangle = \sum_{m\neq n} \frac{\langle m^{(0)}|H'|n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}} |m^{(0)}\rangle$, and
- $E_n^{(2)} = \langle n^{(0)}|H'|n^{(1)}\rangle = \sum_{m\neq n} \frac{\left|\langle m^{(0)}|H'|n^{(0)}\rangle\right|^2}{E_n^{(0)} - E_m^{(1)}}$.